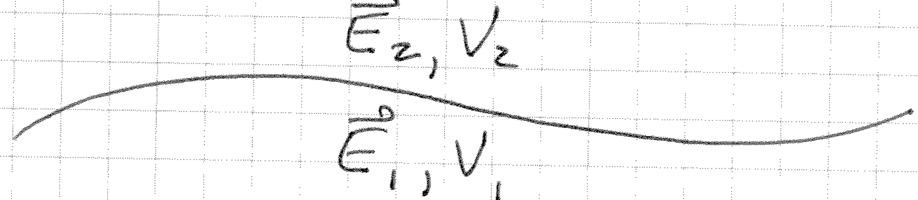
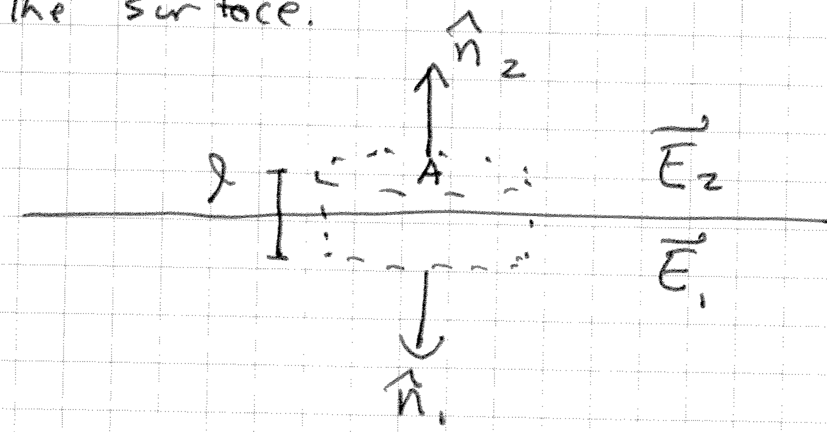
Electrostatic Boundary Conditions

Consider any surface and let $\vec{E}_1$, V be the field and potential on one side of the surface and $\vec{E}_2$, V_2 the field and potential on the other side.



<u>How are $\vec{E}_1, \vec{E}_2$ and V_1, V_2 related?</u>

Use a Gaussian surface, called a <u>Gaussian pillbox</u>, a short cylinder with faces parallel to the surface.



If $l \to 0$, Φ_e out of the sides of the pillbox also $\to 0$.

Apply Gauss' Law

$$\Phi_e = \vec{E}_1 \cdot \hat{n}_1 A + \vec{E}_2 \cdot \hat{n}_2 A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

where σ is the surface charge density in the pillbox.

Let upward normal be $\hat{n}$. $\hat{n} = \hat{n}_2$, $\hat{n} = -\hat{n}_1$

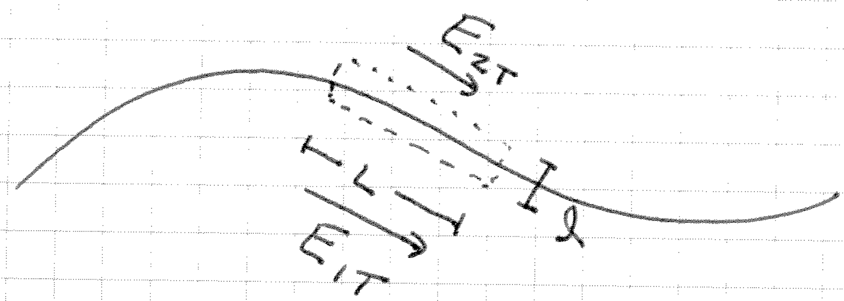
$$\vec{E}_2 \cdot \hat{n} - \vec{E}_1 \cdot \hat{n} = \frac{\sigma}{\varepsilon_0}$$

Let $E_{in} = \vec{E}_i \cdot \hat{n} \equiv$ Component of $\vec{E}$ $\perp$ to the surface

$$\Longrightarrow \quad E_{2n} - E_{1n} = \frac{\sigma}{\varepsilon_0}$$

$\Rightarrow$ The normal component of the field is discontinuous by σ/ε_0

Apply Stoke's Thm   Chose a "Stokesan" loop with sides $\parallel$ to the surface s.t. l becomes small.



Let E_{1T}, E_{2T} be the components of the field tangent to the surface.

$$\text{Since} \quad \nabla \times \vec{E} = 0 \overset{\text{Stokes}}{\Longrightarrow} \oint_C \vec{E} \cdot d\vec{l} = 0$$

$$= E_{1T} L - E_{2T} L$$

$$\Longrightarrow \quad E_{1T} = E_{2T}$$

$\Longrightarrow$ The tangential components of the field are continuous.

$\Longrightarrow$ The Gaussian pillbox and the Stokesian path are two of the best tricks in the game.

The tangential and normal boundry condition can be combined into a single expression

$$\vec{E}_2 - \vec{E}_1 = \frac{\sigma}{\varepsilon_0}\hat{n}$$

where $\hat{n}$ is the normal pointing from 1 to 2.

The potential is the integral of the field so the potential is continuous across the surface.

$$V_1 = V_2$$

$$\vec{E}_1 = -\nabla V_1 \qquad \vec{E}_2 = -\nabla V_2$$

$$\vec{E}_2 - \vec{E}_1 = -\nabla V_2 - (-\nabla V_1) = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$-\hat{n}\cdot\nabla V_2 + \hat{n}\cdot\nabla V_1 = \frac{\sigma}{\varepsilon_0}\hat{n}\cdot\hat{n} = \frac{\sigma}{\varepsilon_0}$$

$\hat{n}\cdot\nabla V = \dfrac{\partial V}{\partial n}$ = The derivative of V normal to the surface

$$\frac{\partial V_2}{\partial n} - \frac{\partial V_1}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

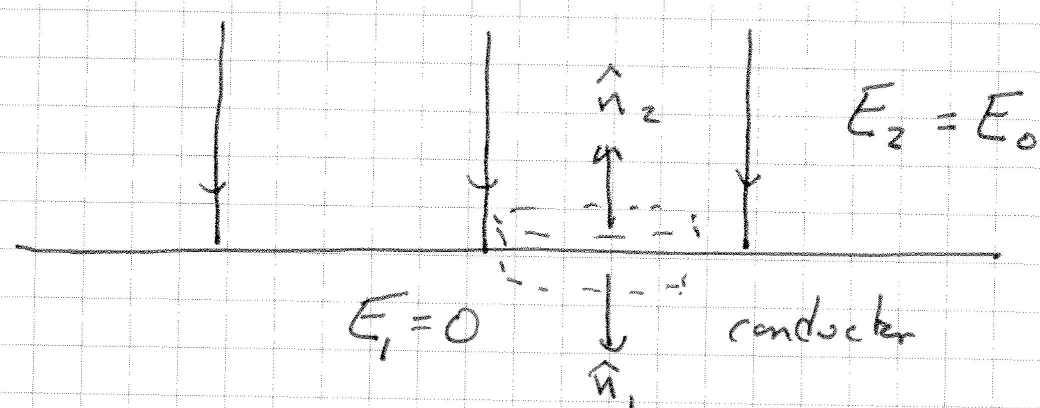
$\Longrightarrow$ The derivative of V changes by $-\sigma/\varepsilon_0$ across the surface.

Note, this relation with the derivative in a direction is general.

$$\frac{\partial V}{\partial r} = \hat{r} \cdot \nabla V$$

$$\frac{\partial V}{\partial x} = \hat{x} \cdot \nabla V$$

<u>Ex</u> Surface charge density at surface of a conductor.



$$\frac{\Phi_e}{A} = \vec{E}_2 \cdot \hat{n}_2 + \vec{E}_1 \cdot \hat{n}_1 = \frac{\sigma}{\varepsilon_0}$$

$$-E_0 + 0 = \frac{\sigma}{\varepsilon_0}$$

$$\boxed{\sigma = -\varepsilon_0 E_0}$$

Note, this makes the pressure $P = \frac{1}{2}\sigma E_0 = \frac{1}{2}\varepsilon_0 E_0^2$ upward.