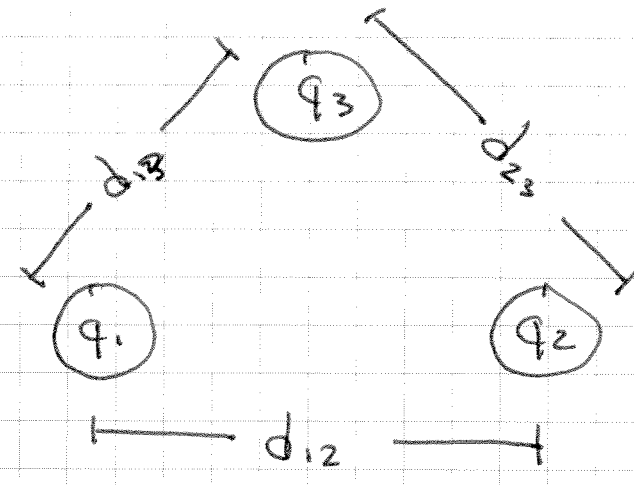
Electrostatic Energy

Consider three point charges



I. <u>Total Energy of System (U)</u> - This would be the energy recovered as KE is the system were allowed to move infinitely ~~far~~ apart OR the total ~~energy~~ work required to build the system.

II. <u>Potential Energy of q_i (U_i)</u> This is the kinetic energy that would be recovered if q_i were released OR the total work required to move q_i from ∞ if $q_{j \neq i}$ were fixed in place.

The potential energy of q_3 is

$$U_3 = q_3 V(\vec{r}_3)$$

$$= q_3\left(\frac{k q_1}{d_{13}} + \frac{k q_2}{d_{23}}\right)$$

If q_3 was released, $\Delta KE = \Delta U$, so the kinetic energy after q_3 has moved very far from the system $(U_f = 0)$ is

$$U_3 = KE_f = \frac{1}{2} m v_f^2$$

The work to move q_3 from ∞ to $\vec{r}_3$ is

$$W = U_3$$

The total energy U of the 3 charge system is the work to build it piece by piece.

The work to place q_1 with no other charges present is 0

$$W_1 = 0$$

The work to place q_2 with q_1 fixed in position is

$$W_2 = q_2 V_1(\vec{r}_2)$$
$$= q_2 \left(\frac{k\, q_1}{d_{12}} \right)$$

The work to place q_3 with q_1 and q_2 in place is

$$W_3 = q_3 V_1(\vec{r}_3) + q_3 V_2(\vec{r}_3)$$
$$= q_3 \left(\frac{k q_1}{d_{13}} \right) + q_3 \left(\frac{k q_2}{d_{23}} \right)$$

The total energy of the system is

$$U = W_1 + W_2 + W_3 = \text{Work total}$$

$$= \frac{k\, q_1 q_2}{d_{12}} + \frac{k\, q_1 q_3}{d_{13}} + \frac{k\, q_2 q_3}{d_{23}}$$

$$= \sum_i \sum_{i<j} \frac{k\, q_i q_j}{d_{ij}}$$

$$U = \frac{1}{2} \sum_i \sum_{j \neq i} \frac{k q_i q_j}{d_{ij}}$$

$$= \frac{1}{2} \sum_i \sum_{j \neq i} q_i V_j(\vec{r}_i)$$

$$= \frac{1}{2} \sum_i q_i V(\vec{r}_i)$$

where $V(\vec{r}_i)$ is the total potential of all charges except q_i.

$$V(\vec{r}_i) = \sum_{i \neq j} \frac{k q_j}{d_{ij}}$$

We can write these sums ~~as~~ for continuous distributions

$$W = U = \frac{1}{2} \int V \, dq$$

$$= \frac{1}{2} \int \rho V \, d\tau$$

We can also write the energy in terms of the field.

$$\nabla \cdot \vec{E} = \rho / \varepsilon_0 \quad \text{Gauss}$$

$$\rho = \varepsilon_0 \nabla \cdot \vec{E}$$

$$U = \frac{1}{2} \int \rho V \, d\tau = \frac{1}{2} \varepsilon_0 \int (\nabla \cdot \vec{E}) V \, d\tau$$

From Griffiths cover,

$$\nabla \cdot (f \vec{A}) = f (\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla f$$

$$f \Rightarrow V \qquad \vec{A} \Rightarrow \vec{E}$$

$$\nabla \cdot (V \vec{E}) = V (\nabla \cdot \vec{E}) + \vec{E} \cdot \nabla V$$

$$= V (\nabla \cdot \vec{E}) - \vec{E} \cdot \vec{E}$$

$$\text{since} \quad \vec{E} = -\nabla V$$

$$U = W = \frac{1}{2} \varepsilon_0 \int V (\nabla \cdot \vec{E}) \, d\tau$$

$$= \frac{1}{2} \varepsilon_0 \int_{\text{Volume}} \nabla \cdot (V \vec{E}) \, d\tau + \frac{1}{2} \varepsilon_0 \int_{\text{Volume}} \vec{E} \cdot \vec{E} \, d\tau$$

Divergence Thm

$$\int_{\text{Volume}} \nabla \cdot (V\vec{E})\, d\tau = \int_{\text{Surface}} V\vec{E} \cdot d\vec{a}$$

Let $S \to \infty$, if charge is compact $\vec{E} \to 0$ as $\frac{1}{r^2}$

and $V \to 0$ as $1/r$ so $\int_S \to 0$

$$W = U = \frac{1}{2}\varepsilon_0 \int_{\text{space}} \vec{E} \cdot \vec{E}\, d\tau$$

Energy Density of Electric Field (u)

$$u = \frac{1}{2}\varepsilon_0 \vec{E} \cdot \vec{E}$$

$\Rightarrow$ If you want the energy of one of these systems of charge, integrate the energy density over all space.