Electrostatics

The full set of 4 Maxwell's equations are complicated coupled differential equations. Before tackling them as a whole, we will look at a number of physically interesting cases that simplify the solution.

A system is <u>electrostatic</u> if all net charge is stationary, that is if $\vec{J}=0$ and ρ is constant in time. If nothing is changing in time, there is no reason for the fields to change with time

so $$\frac{\partial \vec{E}}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \vec{B}}{\partial t} = 0$$

with these simplifications, Maxwell's equations become

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \qquad \nabla \times \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = 0$$

From Helmholtz' Thm, and the assumption that $\vec{B} \to 0$ as $\vec{r} \to \infty$, we find $\vec{B} = 0$ everywhere.

Since $\nabla \times \vec{E} = 0$, $\exists$ a function $V(\vec{r})$ s.t. $\vec{E} = -\nabla V$. This function is called the scalar potential or the electric potential.

The text develops V separately from $\vec{E}$ (somewhat), but I would like to develop the two together.

To develop a physical understanding of $V(\vec{r})$, we need some ~~physics~~ physics not contained in Maxwell's eqns; the effect of electromagnetic fields on changed particles.

<u>Lorentz Force</u> The force $\vec{E}$ and $\vec{B}$ exert on a particle of charge q moving with velocity $\vec{v}$ is

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

For electrostatic situations,

$$\vec{F} = q\vec{E}$$

If the scalar potential had no physical interpretation, it would be an extremely useful computational tool. To compute $\vec{E}$ we need to calculate $E_x(\vec{r})$, $E_y(\vec{r})$, and $E_z(r)$, three functions. To compute $V(\vec{r})$, we only need one function.

Dfn Scalar Potential ($V(\vec{r})$) - Function s.t.

$$\vec{E} = -\nabla V \qquad \text{(electrostatic)}$$

- Units Volt $\quad 1V = 1\frac{Nm}{C} = 1\,J/C$
- Only defined up to an arbitrary constant
- Not the same thing as potential energy.

Poisson's Eqn

$$\nabla \cdot \vec{E} = \rho/\epsilon_0 \implies \nabla \cdot (-\nabla V) = \rho/\epsilon_0$$

$$\nabla^2 V = -\rho/\epsilon_0$$

In charge free ($\rho = 0$) regions, this becomes Laplace's Eqn

$$\nabla^2 V = 0$$

the most studied differential eqn ever.

The electric potential will be related to the energy of a system. To develop the energy of a system of charge, consider the work required to build it.

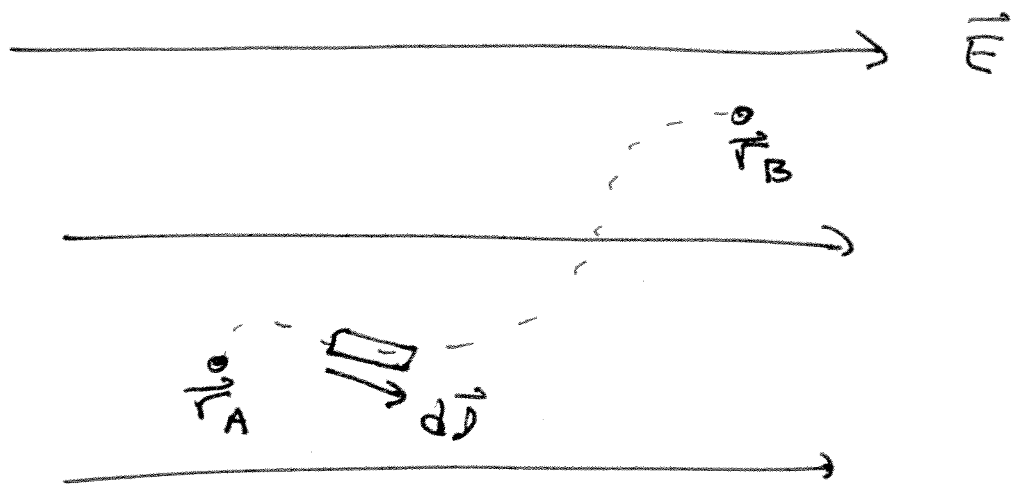
<u>Work</u> The work done by an external agent to move a charged particle through an electric field is defined as

$$W_{AB} = \int_{\vec{r}_A \to \vec{r}_B} \vec{F}_{ext} \cdot d\vec{l}$$

$\vec{r}_A$ beginning point, $\vec{r}_B$ ending point

$\vec{F}_{ext}$ - Force applied by external agent

$d\vec{l}$ - element of path the particle is moved along.



If the external agent moves the charged particle in a way s.t. the kinetic energy of the particle does not change, the work done will equal the change in the <u>potential energy</u> ΔU of the charge field system.

$$W_{AB} = \Delta U_{AB}$$

To move the charged particle with changing the kinetic energy, the force applied must just balance the force exerted by the field

$$\vec{F}_{ext} = -Q\vec{E}$$

$$W_{AB} = -Q \int_{\vec{r}_A \to \vec{r}_B} \vec{E} \cdot d\vec{l} = \Delta U_{AB}$$

if $\Delta KE_{AB} = 0$

$$W_{AB} = -Q \int_{\vec{r}_A \to \vec{r}_B} -\nabla V \cdot d\vec{l}$$

$$= Q \int_{\vec{r}_A \to r_B} dV$$

$$= Q\left(V(\vec{r}_B) - V(\vec{r}_A)\right)$$

(gradient thm)

<u>Dfn Electric Potential Energy</u> $(U(\vec{r}))$

The difference in potential energy $U_{o\vec{r}}$ between a reference point $\vec{r}_o$ which we will define to have zero potential energy and the point $\vec{r}$

$$U(\vec{r}) = W_{o\vec{r}} = \Delta U_{o\vec{r}}$$

TOPS.

$$W_{AB} = U(\vec{r}_B) - U(\vec{r}_A) = QV(\vec{r}_B) - QV(\vec{r}_A)$$

If we choose the same $\vec{r}_o$ for V, then

$$U(\vec{r}) = Q\,V(\vec{r})$$

Dfn <u>Electric Potential Difference</u> (ΔV_{AB}).

The work per unit charge to move a charge from A to B.

$$\Delta V_{AB} = V(\vec{r}_B) - V(\vec{r}_A)$$

$$= -\int_{\vec{r}_A \to \vec{r}_B} \vec{E} \cdot d\vec{\ell}$$

The potential $V(\vec{r})$ is then the potential difference with respect to reference point $\vec{r}_o$

$$V(\vec{r}) = \Delta V_{o\vec{r}}$$

$\Rightarrow$ It is traditional to choose $\vec{r}_o$ at ∞ so the potential at ∞ is zero.

$\Rightarrow$ There are some systems where this choice does not work.

We will build fields of complicated systems out of fields of simpler systems.

Principle of Linear Superposition

The total field is the vector sum of the fields of the individual charges.

$\Rightarrow$ The total potential is the ~~vec~~ sum of the potentials of the individual charges

Proof Suppose a charge distribution ρ can be divided into two pieces, $\rho = \rho_1 + \rho_2$ and the field of each piece determined independently s.t.

$$\nabla \cdot \vec{E}_1 = \rho_1/\epsilon_0 \qquad \nabla \cdot \vec{E}_2 = \rho_2/\epsilon_0$$

$$\nabla \times \vec{E}_1 = 0 \qquad \nabla \times \vec{E}_2 = 0$$

– or –

$$\nabla^2 V_1 = -\rho_1/\epsilon_0 \qquad \nabla^2 V_2 = -\rho_2/\epsilon_0$$

The field of the total distribution must solve

$$\nabla \cdot \vec{E} = \rho/\epsilon_0 \qquad \nabla \times \vec{E} = 0$$

– or –

$$\nabla^2 V = -\rho/\epsilon_0$$

Evidently, $\vec{E} = \vec{E}_1 + \vec{E}_2$ or $V = V_1 + V_2$ will be a solution for the full field.

Note, $\nabla \times \vec{E} = 0$ guarontees that ΔV is independant of path