Electromotive Force

Consider the work done by the electromagnetic fields as a charged particle moves along the path C

$$\text{Work} = \int_C \left(\vec{F}_e + \vec{F}_m\right) \cdot d\vec{l}$$

$$= q \int_C \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot d\vec{l}$$

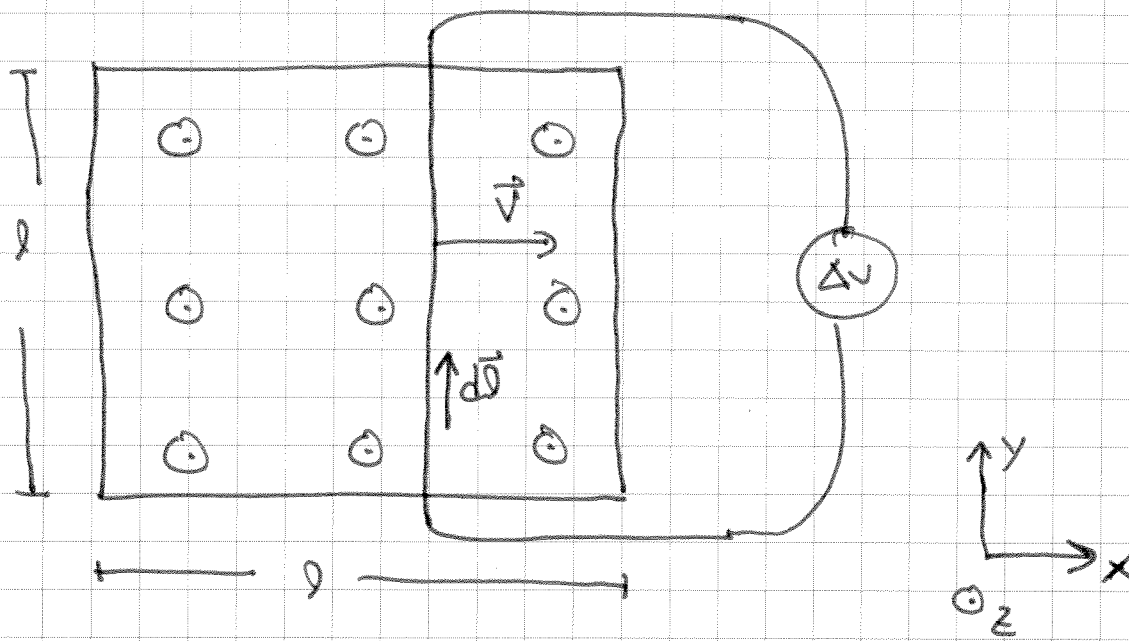
Electromotive Force (emf) – Work per unit charge done by the electromagnetic fields as a charged particle moves along the path C.

$$\text{emf} = \frac{W}{q} = \int_C \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot d\vec{l}$$

Motional EMF The part of the emf resulting from the motion of the current path.

$$\text{emf} = \int_C \left(\vec{v} \times \vec{B}\right) \cdot d\vec{l}$$

Ex Pull wire through magnetic field



$$\vec{B} = B_0 \hat{z} \qquad \vec{v} = v_0 \hat{x} \qquad d\vec{l} = dy\, \hat{y}$$

$$\vec{v} \times \vec{B} = v_0 B_0 (\hat{x} \times \hat{z}) = -\hat{y} v_0 B_0$$

$$(\vec{v} \times \vec{B}) \cdot d\vec{l} = -v_0 B_0 \hat{y} \cdot \hat{y}\, dy = -v_0 B_0 dy$$

Motional EMF

$$\text{emf} = \int_c (\vec{v} \times \vec{B}) \cdot d\vec{l} = -v_0 B_0 l$$

$\Rightarrow$ We would measure this in Volts on a voltmeter.

Note, the directions are crucial, the magnetic force must be along the wire.

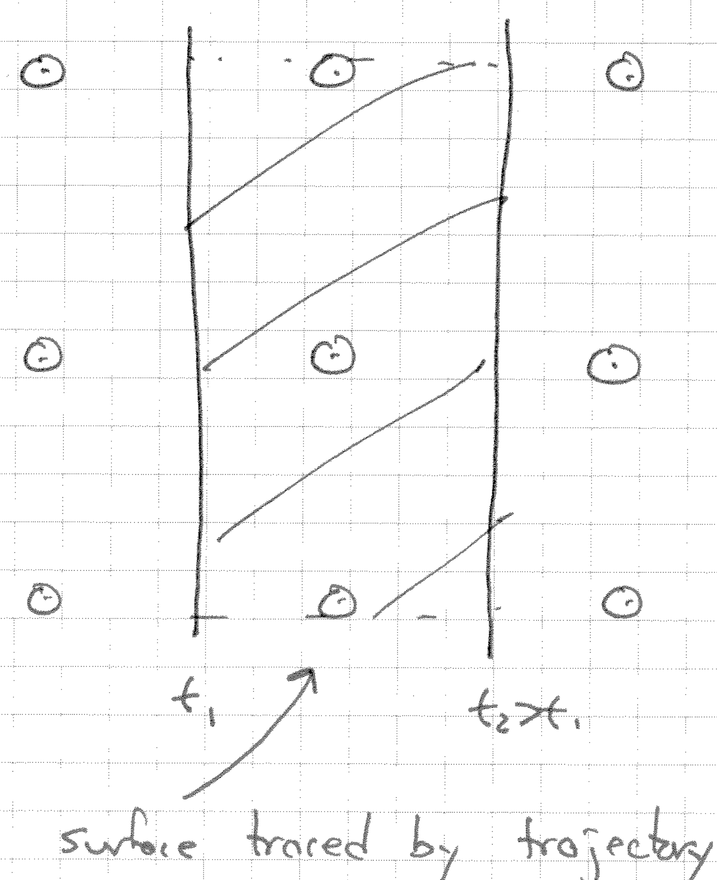
(1) If $\vec{v}$ out of page, $\vec{v} \times \vec{B} = 0$, emf $= 0$.

(2) If $\vec{v}$ toward the top of the page,

$$\vec{v} = v_0 \hat{y} \implies \vec{v} \times \vec{B} = v_0 B_0 \hat{x}$$

$$\implies (\vec{v} \times \vec{B}) \cdot d\vec{l} = 0.$$

$\implies$ The key to having a non-zero emf is for the surface formed by the trajectory to have a non-zero flux through the surface.



Magnetic Flux Φ_m

The magnetic flux through surface S is

$$\Phi_m = \int_S \vec{B} \cdot d\vec{a}$$

$\Rightarrow$ If field is uniform and surface is flat,

$$\Phi_m = (\vec{B} \cdot \hat{n})\, A = BA\cos\theta$$

area of surface

θ - angle between $\vec{B}$ and $\hat{n}$.

For our wire moving in the field at constant velocity, the flux traced out by the surface from time t_0 to time t is

$$\Phi_m = B_0 \ell v_0 (t - t_0)$$

The emf is then the time rate of change of the flux

$$\text{emf} = -\frac{d\Phi_m}{dt} = -B_0 \ell v_0$$

<u>Ex</u> How many volts? Suppose $B_0 = \frac{1}{4}\,T$

$V_0 = 10\,m/s$, $l = 10\,cm$

$$\text{emf} = B_0 v_0 l = \frac{1}{4}\,V$$

Multiple Turns (N) - We will often deal with flux through surfaces formed by multiple turns of wire, in this case the flux becomes

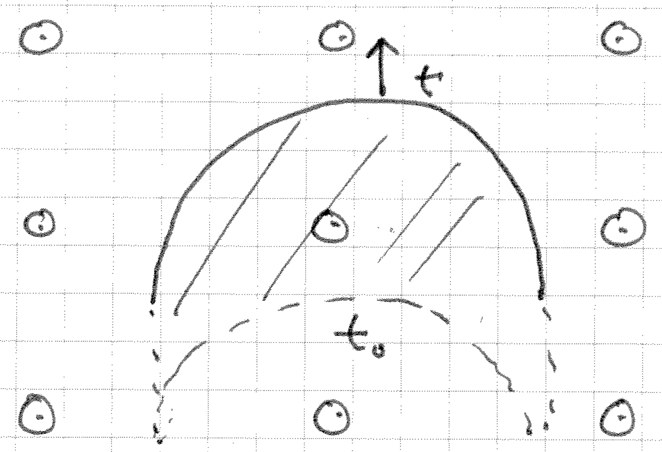
$$\Phi_m = N \int_S \vec{B} \cdot d\vec{a}$$

<u>Flux Rule</u> - The emf around a closed curve C (or a curve that can be completed) is proportional to the change in flux through the surface S bounded by C.

$$\text{emf} = -\frac{d\Phi_m}{dt}$$

⇒ The flux rule is more powerful than the emf integral.

Ex Compute emf between ends of half circle moving with velocity v_0 through constant field B_0.



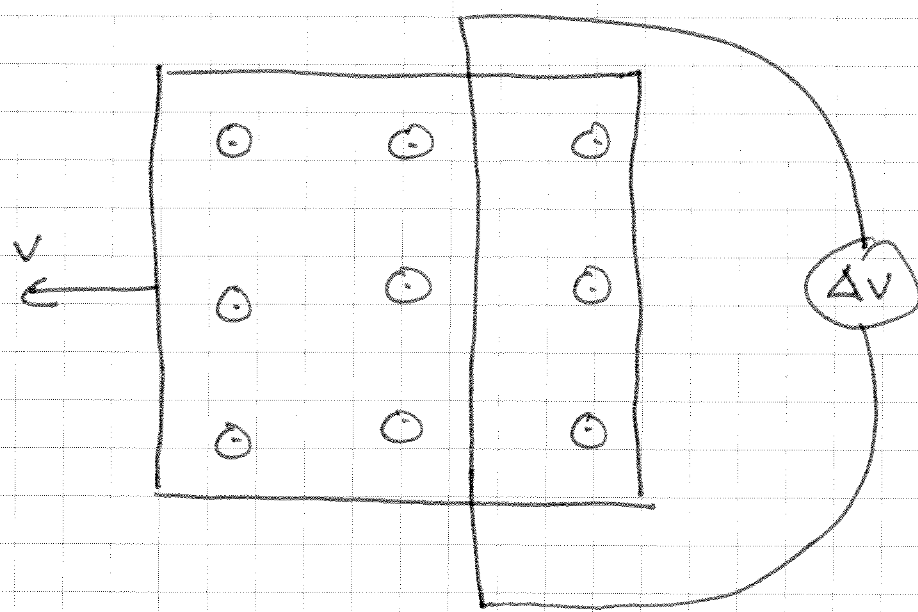
The flux through the surface above is

$$\Phi_m = B_0 \, 2R \, v_0 (t - t_0)$$

The emf using the flux rule

$$emf = \frac{-d\Phi_m}{dt} = -2R v_0 B_0$$

Now consider the opposite experiment with our moving wire; move the magnet instead



$\Rightarrow$ Relativity says we should measure same ΔV.

$$\text{emf} = \int_C (\vec{E} + \underbrace{\vec{v} \times \vec{B}}_{=0}) \cdot d\vec{\ell}$$

$\Rightarrow \vec{E} \neq 0$

$\Rightarrow$ A changing magnetic field produces an electric field.

Faraday's Law The integral of the electric field around a closed path C is proportional to the time rate of change of the magnetic flux through a surface S bounded by C.

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\int_S \vec{B} \cdot \hat{n}\, da$$

⟹ The path C is fixed.

⟹ The positive normal of S is given by the normal right hand convention

⟹ Special case of flux rule.

⟹ The direction of the electric field and therefore the direction of induced current if C follows a conducting path follows direction from the convention for the positive normal.

Sorting out the normal directions gives Lenz' Law

Lenz' Law - The induced current flows in a direction s.t. the flux of the induced current through S opposes the changing flux through S.

Differential Form

$$\oint_C \vec{E} \cdot d\vec{l} \overset{\text{Stokes}}{=} \int_S (\nabla \times \vec{E}) \cdot d\vec{a}$$

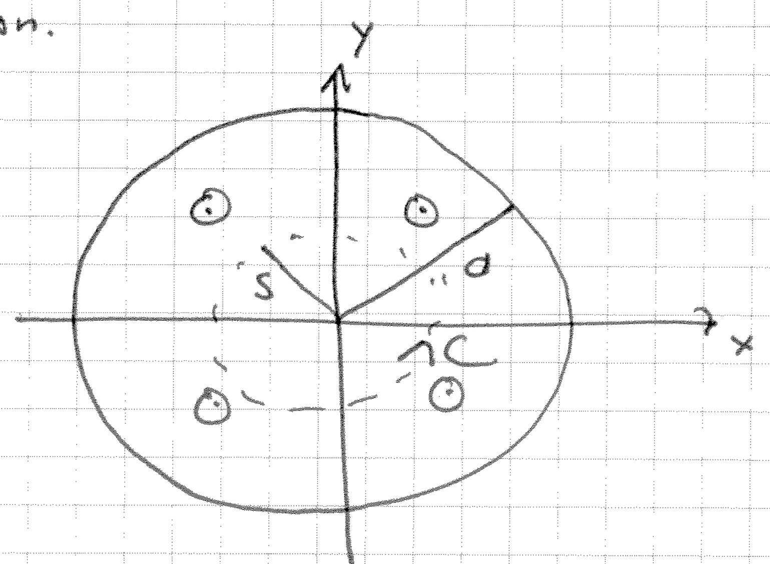
$$= -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{a}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{Faraday's Law}$$

$\Rightarrow$ Correct everywhere without restriction.

$\Rightarrow$ Does not require a loop of wire.

<u>Ex</u> Consider a cylindrical region $s<a$ containing a magnetic field $\vec{B} = B_0 \frac{t}{T} \hat{z}$ where T is constant. Compute the electric field in the region.



B increasing in strength out of page

$\Rightarrow$ By RHR, positive normal to S out of page

$\Rightarrow$ The flux through a surface of radius s is

$$\Phi_m = \oint \vec{B} \cdot d\vec{a} = \pi s^2 B$$

$$= \pi s^2 B_0 \, t/T$$

Note sign because $\hat{n} = \hat{z}$.

<u>Faraday's Law</u>

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_m}{dt} = \frac{-\pi s^2 B_0}{T}$$

By symmetry, the electric field must be circular about the z axis, $\vec{E} = E(s)\hat{\phi}$.

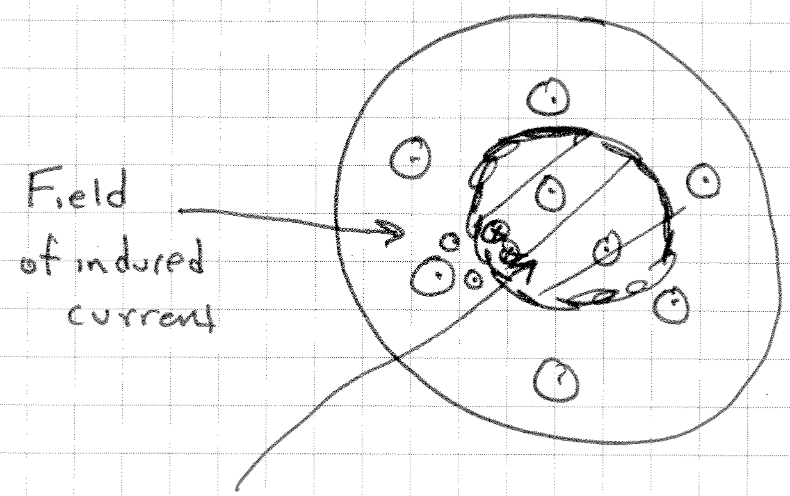
Note ~~at~~ our chosen path has $d\vec{l} = s\, d\phi\, \hat{\phi}$

$$\oint \vec{E} \cdot d\vec{l} = \int_0^{2\pi} s\, d\phi\, E(s) = 2\pi s E$$

$$= -\frac{d\Phi_m}{dt} = -\frac{\pi s^2 B_0}{T}$$

$$\vec{E} = \frac{-sB_0}{2\tau}\,\hat{\phi}$$

⇒ Check direction using Lenz' Law. The electric field must be in the direction of the current that would flow in a conducting loop placed in the ~~electric~~ field.



Flux increasing out of the page, so induced current produces flux into the page, induced current must flow clockwise by RHR.

Clockwise $= -\hat{\phi}$ ✓