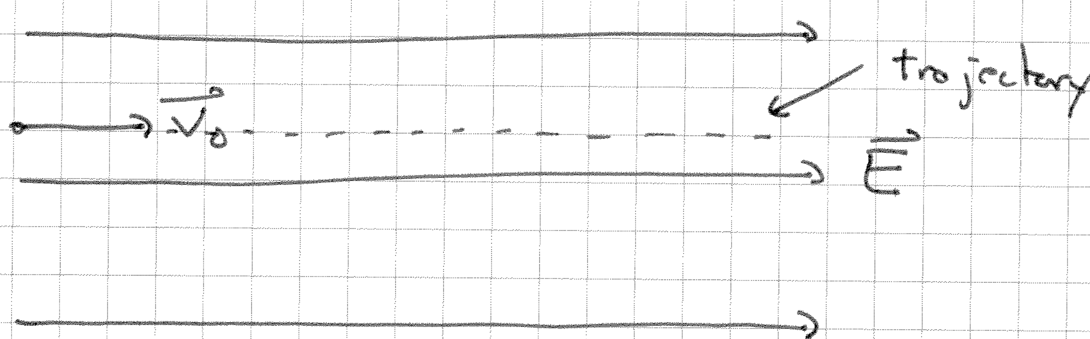
Electromagnetic Force

The electromagnetic force on a charged particle q moving at velocity $\vec{v}$ is

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Electric Force - Take $\vec{E} = E_0 \hat{x}$

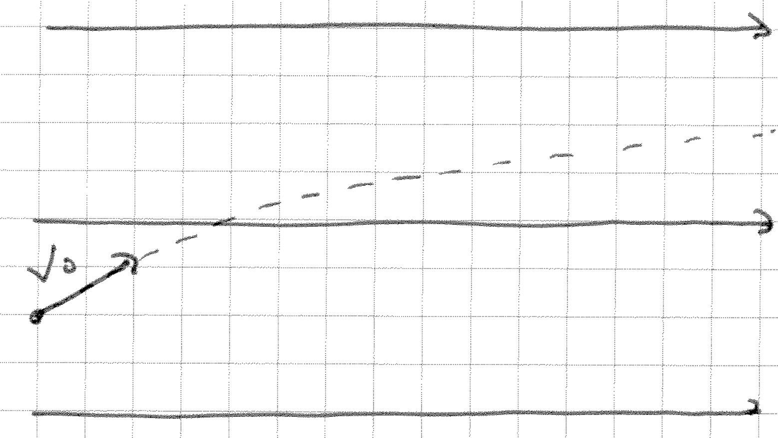


If $\vec{v}_0 = v_0 \hat{x}$, $x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$

$$F = qE = ma$$

$$a = \frac{qE}{m}$$

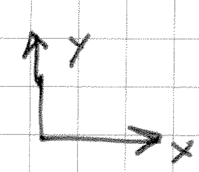
If $\vec{v}_0$ is not $\parallel$ to $\vec{E}$, $\vec{v}_0 = (v_{0x}, v_{0y}, 0)$



$$x(t) = x_0 + v_{0x} t + \frac{1}{2} a t^2$$

$$y(t) = y_0 + v_{0y} t$$

If we take $\vec{E} = -E_0 \hat{y}$



$$x(t) = x_0 + v_{0x} t$$

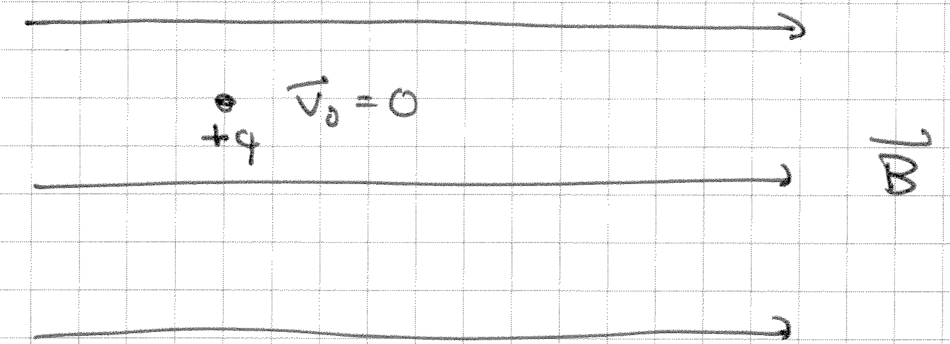
$$y(t) = y_0 + v_{0y} t - \frac{1}{2} a_y t^2$$

$$a_y = \frac{q E_0}{m}$$

$\Rightarrow$ Force changes direction if $q \to -q$.

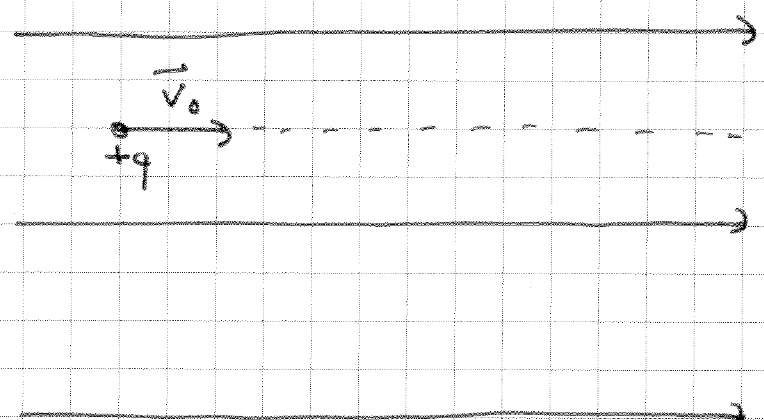
Now consider a magnetic field $\vec{B} = B_0 \hat{x}$

If particle with charge q is released ($\vec{v}_0 = 0$) in the field



The particle just sits there, $\vec{v} = 0 \Rightarrow \vec{F}_m = 0$

Now suppose particle has initial velocity $\vec{v}_0 = v_0 \hat{x}$

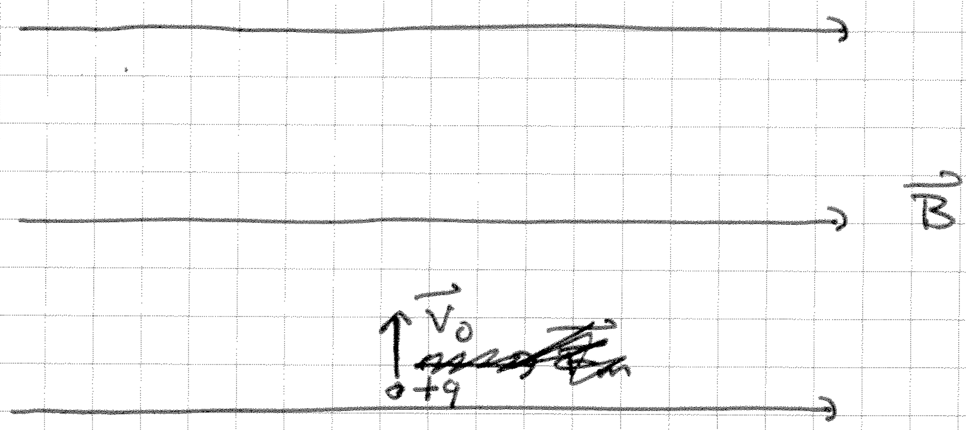


$$\vec{F}_m = q\vec{v} \times \vec{B} = q v_0 \hat{x} \times B_0 \hat{x} = 0$$

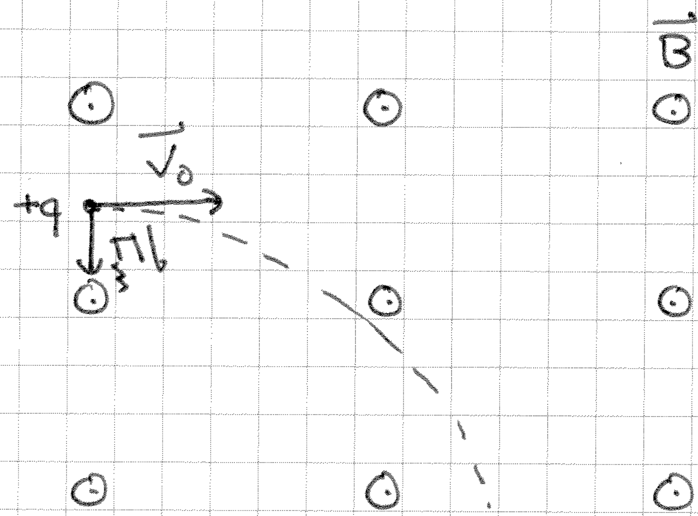
$$y(t) = 0 \qquad x(t) = \cancel{x_0}\ x_0 + v_0 t$$

No acceleration

Now suppose $\vec{v}_0 \perp \vec{B}$



The initial force is into the page. Change to orientation of the drawing so that $\vec{B}$ points out of the paper.



Since $\vec{F} = q\vec{v} \times \vec{B}$, $\vec{F} \perp \vec{v}$, $\vec{F} \perp \frac{d\vec{r}}{dt}$

$\Rightarrow \vec{F} \perp d\vec{r}$

but work Work $= \int \vec{F} \cdot d\vec{r} = 0$

$\Rightarrow$ The magnetic field does no work.

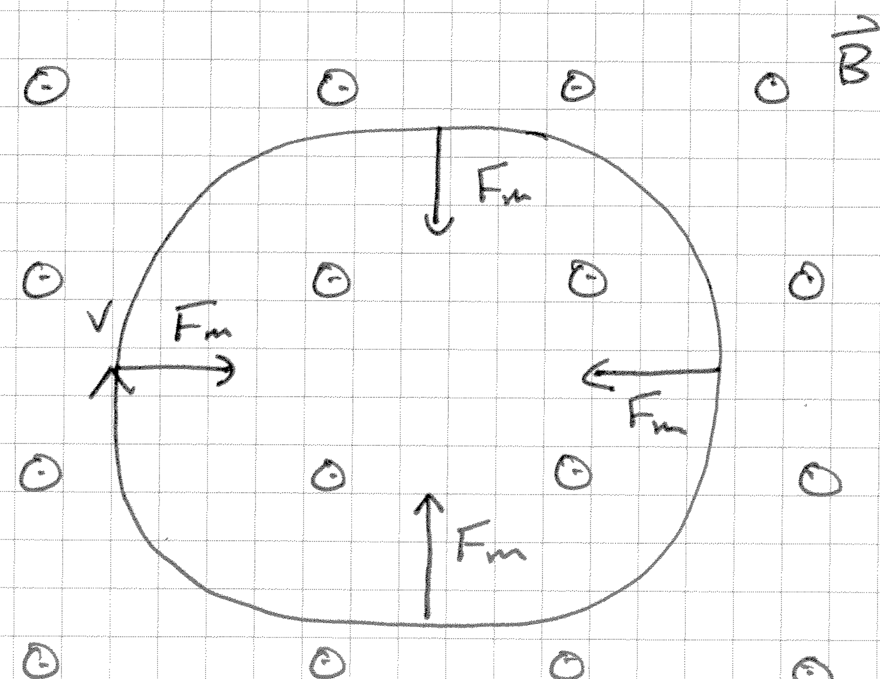
$\Rightarrow$ The magnetic field cannot change the energy of the particle.

$\Rightarrow$ Magnetic field can change direction but not speed.

For the particle we are considering in the second view, the force is in the plane of the page, so $\vec{B} \perp \vec{v}$

$\Rightarrow \quad |\vec{F}_m| = |q\vec{v} \times \vec{B}| = qvB_0 = \text{constant}$

The particle moves under the influence of a constant magnitude force of constant magnitude that acts $\perp$ to its velocity



$\Rightarrow$ The particle moves in a circle!

The acceleration of anything moving in a circle is

$$a_c = \frac{v^2}{r} \equiv \text{centripetal acceleration}$$

This can also be written in terms of the angular velocity ω, $v = \omega r$

$$a_c = r\omega^2$$

$$\omega = \frac{d\phi}{dt}$$

$\Longrightarrow$ Note, ω constant because kinetic energy is constant.

<u>Newton II</u>

$$F_m = m a_c = \frac{mv^2}{r} = mr\omega^2$$

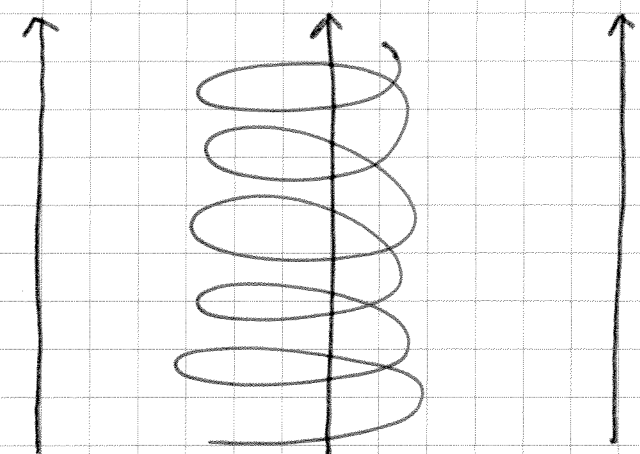
$$= qvB = q\omega r B$$

$$\Longrightarrow r = \frac{mv}{qB} \quad \text{or} \quad \omega = \frac{qB}{m}$$

<u>Cyclotron Frequency</u> $\quad \omega = \frac{qB}{m}$

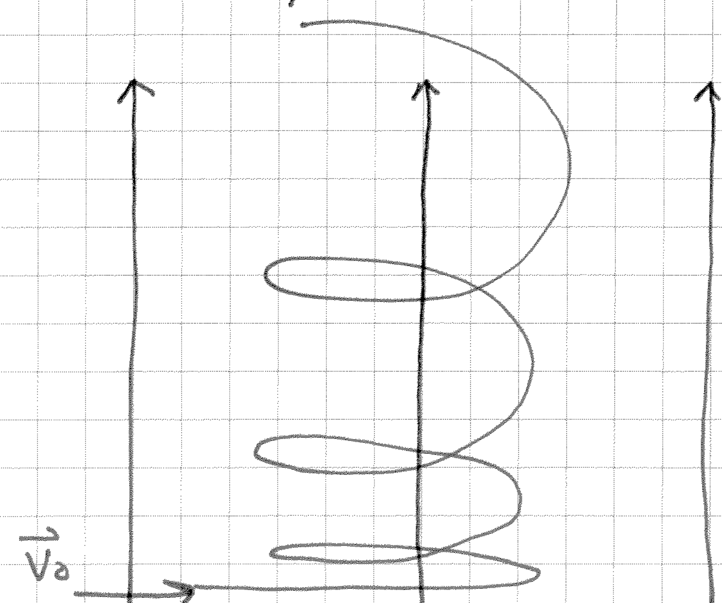
Suppose the particle has some initial velocity component ∥ to B. That component of velocity is not changed by B and the particle travels in a helix

spirals equally spaced



$\vec{B}$ = constant $\quad \vec{E} = 0$

Suppose now there is an electric field $\vec{E}$ parallel to magnetic field $\vec{B}$ and the particle has initial velocity $\perp$ to $\vec{B}$



Travels in helix with increasing spacing between wraps.

$\vec{B} = B_0 \hat{y} \qquad \vec{E} = E_0 \hat{y}$

Let's return to the simpler problem with only a constant magnetic field and do the mechanics properly.

Let $\vec{B} = B_0 \hat{z}$, $\vec{r}_0 = (\cancel{x_0}, 0, 0)$, $\vec{v}_0 = (0, v_0, 0)$

<u>Newton's II Law</u>

$$\vec{F} = q \vec{v} \times \vec{B} = m\vec{a} = m \frac{d^2 \vec{r}}{dT^2}$$

$$\vec{F} = q \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B_0 \end{vmatrix}$$

$$= q B_0 v_y \hat{x} - q B_0 v_x \hat{y}$$

$\Rightarrow$ Note, no force in z-direction so

$$z(t) = z_0 + v_{0z} t$$

Equations of Motion (EOM)

$$\vec{F} = m\vec{a}$$

$$m a_x = q v_y B_o = m \frac{dv_x}{dt}$$

$$m a_y = -q v_x B_o = m \frac{dv_y}{dt}$$

Using $\omega = \frac{qB_o}{m}$

$$\frac{dv_x}{dt} = \omega v_y \qquad \frac{dv_y}{dt} = -\omega v_x$$

Differentiate

$$\frac{d^2 v_x}{dt^2} = \omega \frac{dv_y}{dt} \qquad \frac{d^2 v_y}{dt^2} = -\omega \frac{dv_x}{dt}$$

Substitute

$$\frac{d^2 v_x}{dt^2} = -\omega^2 v_x \qquad \frac{d^2 v_y}{dt^2} = -\omega^2 v_y$$

Solve Both equations are simple harmonic oscillators. Solutions

$$V_x(t) = A_1 \cos\omega t + A_2 \sin\omega t$$

$$V_y(t) = C_1 \cos\omega t + C_2 \sin\omega t$$

Use original eqn

$$\frac{dV_x}{dt} = -A_1\omega \sin\omega t + A_2\omega\cos\omega t$$

$$= \omega V_y = C_1\omega\cos\omega t + C_2\omega\sin\omega t$$

$$C_2 = -A_1 \qquad C_1 = A_2$$

$$V_x(t) = A_1\cos\omega t + A_2\sin\omega t$$

$$V_y(t) = +A_2\cos\omega t - A_1\sin\omega t$$

Initial Conditions

$$V_x(0) = 0 \implies A_1 = 0$$

$$V_y(0) = V_0 \implies A_2 = V_0$$

$$V_x(t) = V_0\sin\omega t \qquad V_y(t) = V_0\cos\omega t$$

Integrate

$$x(t) = -\frac{v_0}{\omega} \cos\omega t + C_3$$

$$y(t) = \frac{v_0}{\omega} \sin\omega t + C_4$$

Initial Conditions

$$x(0) = x_0 = -\frac{v_0}{\omega} + C_3$$

$$C_3 = x_0 + \frac{v_0}{\omega}$$

$$y(0) = 0 = C_4$$

$$x(t) = -\frac{v_0}{\omega} \cos\omega t + x_0 + \frac{v_0}{\omega}$$

$$y(t) = \frac{v_0}{\omega} \sin\omega t$$

Let $x'(t) = x(t) - \left(x_0 + \frac{v_0}{\omega}\right)$

$$x'^2 + y^2 = \left(\frac{v_0}{\omega}\right)^2 \cos^2\omega t + \left(\frac{v_0}{\omega}\right)^2 \sin^2\omega t$$

$$= \left(\frac{v_0}{\omega}\right)^2 = \text{constant}$$

The motion is a circle of radius $R \equiv \frac{v_0}{\omega}$ centered at $x_0 + R$

$$x(t) = -R\cos\omega t + x_0 + R$$

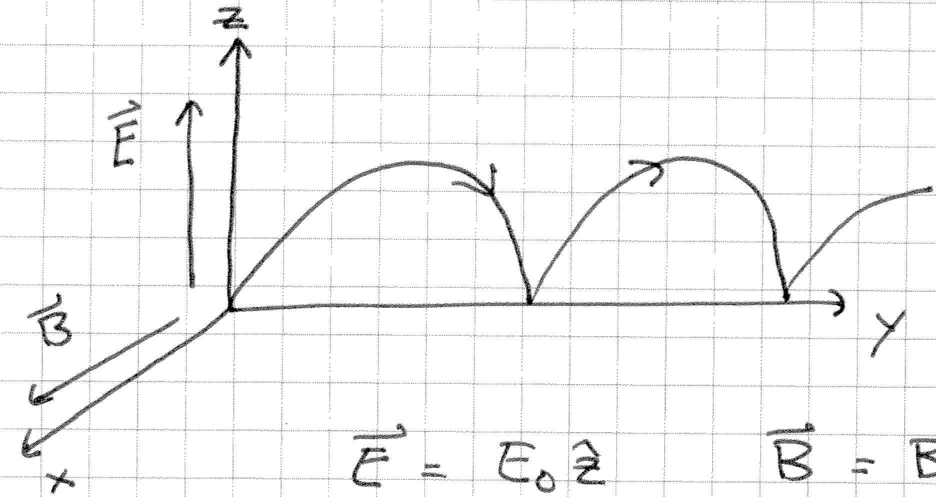
$$y(t) = R\sin\omega t$$

$\Rightarrow$ If an electric field $E_0\hat{z}$ is applied parallel to the magnetic field

$$z(t) = z_0 + v_{0z}t + \frac{1}{2}\frac{qE}{m}t^2$$

Ex Now try crossed $\vec{E}$ and $\vec{B}$ fields. Something weird happens.

If the particle is released from rest



$$\vec{E} = E_0 \hat{z} \qquad \vec{B} = B_0 \hat{x}$$

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & v_y & v_z \\ B_0 & 0 & 0 \end{vmatrix}$$

$$= v_z B_0 \hat{y} - v_y B_0 \hat{z}$$

If particle released, nothing to produce motion in x direction.

Total Force

$$\vec{F} = qE_0\hat{z} - qv_yB_0\hat{z} + qv_zB_0\hat{y}$$

$$= m\vec{a}$$

$$\vec{a} = \left(\frac{qE_0}{m} - v_y\omega\right)\hat{z} + \omega v_z\hat{y}$$

$$= \omega\left(\frac{E_0}{B_0} - v_y\right)\hat{z} + \omega v_z\hat{y} \qquad \omega = \frac{qB}{m}$$

Equations of Motion

$$a_x = 0$$

$$a_y = \frac{dv_y}{dt} = \omega v_z$$

$$a_z = \frac{dv_z}{dt} = -\omega v_y + \omega\frac{E_0}{B_0}$$

Use the same trick again

$$\frac{d^2 v_y}{dt^2} = \omega \frac{dv_z}{dt} = \omega\left(-\omega v_y + \omega \frac{E_0}{B_0}\right)$$

$$= -\omega^2 v_y + \frac{\omega^2 E_0}{B_0}$$

$$\frac{d^2 v_z}{dt^2} = -\omega \frac{dv_y}{dt} = -\omega^2 v_z$$

<u>Solutions</u>

$$v_y(t) = A\sin\omega t + B\cos\omega t + \frac{E_0}{B_0}$$

$$v_z(t) = C\sin\omega t + D\cos\omega t$$

<u>Use original equations</u>

$$\frac{dv_y}{dt} = A\omega\cos\omega t - B\omega\sin\omega t$$

$$= \omega v_z$$

$$= C\omega\sin\omega t + D\omega\cos\omega t$$

$$\Longrightarrow \quad A = D, \quad C = -B$$

$$v_y(t) = A\sin\omega t + B\cos\omega t + E_0/B_0$$

$$v_z(t) = -B\sin\omega t + A\cos\omega t$$

Integrote

$$y(t) = -\frac{A}{\omega}\cos\omega t + \frac{B}{\omega}\sin\omega t + \frac{E_0}{B_0}t + C_y$$

$$z(t) = \frac{B}{\omega}\cos\omega t + \frac{A}{\omega}\sin\omega t$$

Initial Conditions

Release $\vec{V_0} = 0$

$$V_y(0) \Longrightarrow \quad B + \frac{E_0}{B_0} = 0$$

$$B = -\frac{E_0}{B_0}$$

$$V_z(0) = A = 0$$

Therefore

$$y(t) = -\frac{E_0}{\omega B_0}\sin\omega t + \frac{E_0}{B_0}t + C_y$$

$$z(t) = -\frac{E_0}{\omega B_0}\cos\omega t + C_z$$

Initial Condition Released at origin

$$\vec{r}(0) = 0$$

$$y(0) = 0 \Rightarrow C_y = 0$$

$$z(0) = 0 \Rightarrow -\frac{E_0}{\omega B_0} + C_z = 0$$

$$C_z = +\frac{E_0}{\omega B_0}$$

So

$$y(t) = -\frac{E_0}{\omega B_0} \sin\omega t + \frac{E_0}{B_0} t$$

$$z(t) = -\frac{E_0}{\omega B_0} \cos\omega t + \frac{E_0}{\omega B_0}$$

Simplify

$$y(t) = \frac{E_0}{\omega B_0}\left(\omega t - \sin\omega t\right)$$

$$z(t) = \frac{E_0}{\omega B_0}\left(1 - \cos\omega t\right)$$

Let $R = \dfrac{E_0}{\omega B_0}$

$$\left(y(t) - R\omega t\right)^2 + \left(z(t) - R\right)^2$$

$$= R^2\cos^2\omega t + R^2\sin^2\omega t = R^2$$

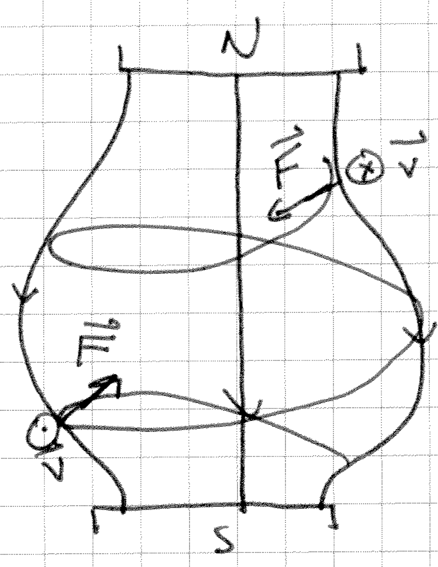
$\Longrightarrow$ Equation of a circle with moving center

$$\vec{r}_{center} = \left(0, R\omega t, R\right)$$

$\Longrightarrow$ Velocity of Center

$$\vec{V}_c = R\omega\,\hat{y} = \frac{E_0}{B_0}\,\hat{y}$$

Magnetic Bottle



Field with gradient forces particles to center