Gauss' Law - Integral Form

Gauss' Law $\quad \nabla \cdot \vec{E} = \rho/\varepsilon_0$

Integrate over volume V

$$\int_V \nabla \cdot \vec{E}\, d\tau = \frac{1}{\varepsilon_0} \int_V \rho\, d\tau$$

$$\shortparallel$$

$$\Phi_e = \oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

Q_{enc} - The total charge in the volume V.

$\Phi_e \equiv$ Electric Flux through surface S $= \int_S \vec{E} \cdot d\vec{a}$

$\Longrightarrow$ If $\vec{E} \parallel \hat{n}$, where $\hat{n}$ is the surface normal, at all points on the surface,

$$\Phi_e = \int_S |\vec{E}|\, da$$

If $|\vec{E}|$ is constant on the surface,

$$\Phi_e = |E| \int_S da = |E| A_s$$

where A_s is the surface area of S.

$\Rightarrow$ The surface normal ($\hat{n}$) is the normal pointing out of the surface for a closed surface.

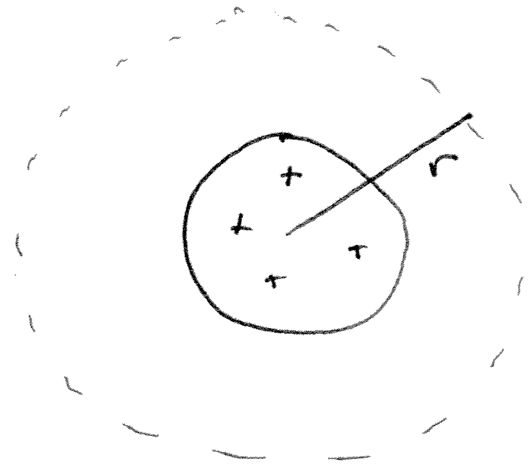
$\Rightarrow$ If the electric field is $\perp$ to $\hat{n}$ at all points on S, $\Phi_e = 0$.

<u>Dfn Gaussian Surface</u> A surface s.t. $\vec{E} \perp \hat{n}$ or $\vec{E} \parallel \hat{n}$ and $|\vec{E}|$ is constant.

$\Rightarrow$ For a Gaussian surface $\Phi_e = |\vec{E}| A_s$ where A_s is the part of the surface where $\vec{E} \parallel \hat{n}$.

$\Rightarrow$ Remember $\Phi_e > 0$ if the field points out of the surface and $\Phi_e < 0$ if the field points into the surface.

Spherically Symmetric Systems

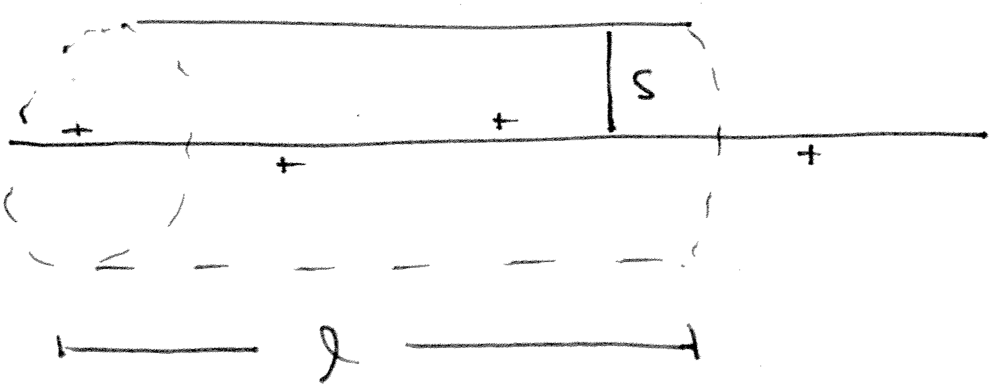


$\Rightarrow$ Use spherical Gaussian Surface

$$\Phi_e = 4\pi r^2 E = E A_s = \frac{Q_{enc}}{\varepsilon_0}$$

Cylindrically Symmetric Systems

\- Must be infinitely long. Use a cylindrical surface of length l and radius s.

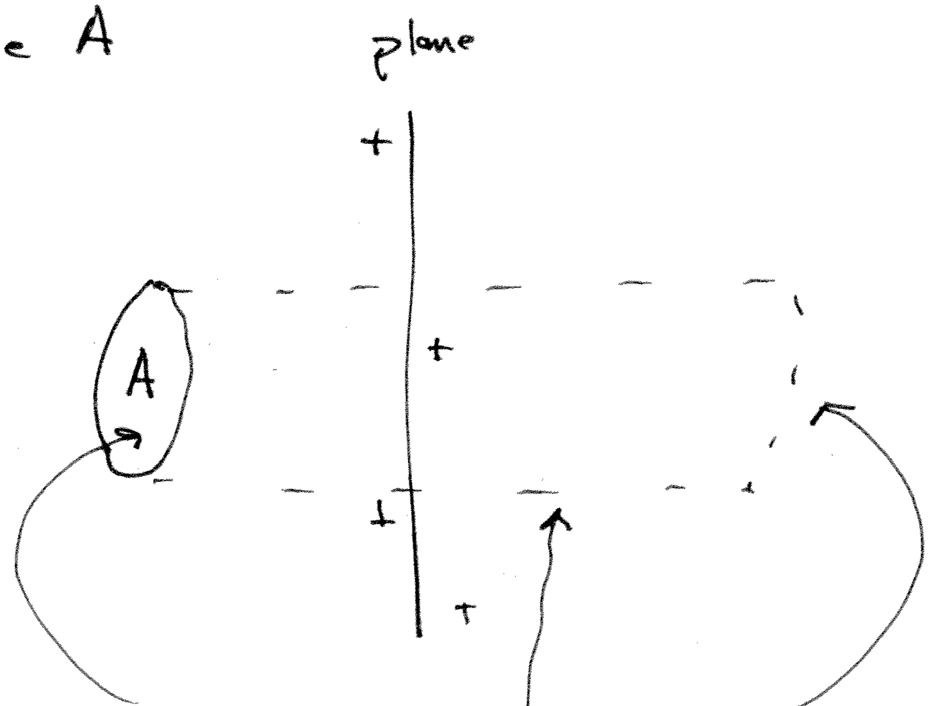


Φ_e ends $= 0$ $\quad\quad$ $\Phi_e = E_{sides} A_{sides} = 2\pi s l E$

$$\Phi_e = 2\pi s \ell E(s) = Q_{enc}/\varepsilon_0$$

Planar Systems (translationally symmetric)

Use a cylindrical surface with ends ∥ to the plane. Let the area of one end of the surface be A



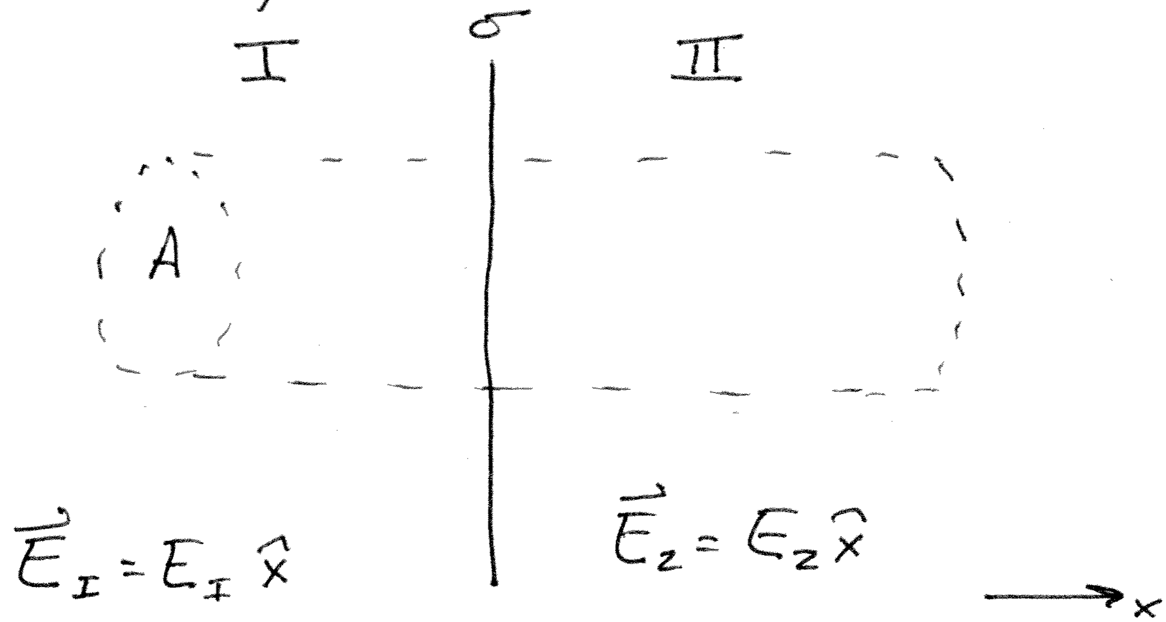
$$\Phi_e = \Phi_{e,\ell} + \Phi_{e,\text{sides}} + \Phi_{e,r}$$

$\Phi_{e,\text{sides}} = 0$ because $\vec{E} \perp \hat{n}$

$$\Phi_e = \pm E_\ell A \pm E_r A = \frac{Q_{enc}}{\varepsilon_0}$$

$\pm$ set by whether flux points out of surface + or into surface −.

Ex Compute field of Infinite Plane with charge density σ.



Method I - Let the math do the work

$$\Phi_e = \Phi_{e,l} + \Phi_{e,r}$$

$$= -E_I A + E_2 A = Q_{enc}/\varepsilon_0$$

If $E_I > 0$, the field points into the surface and the flux is negative, so $-E_I A$ is correctly negative

If $E_I < 0$, the field points out of the surface at the left side and the flux is positive so $-E_I A$ is also correct.

$\Rightarrow$ Check that $E_{II} A = \Phi_{e,r}$ is correct in both cases.

The charge enclosed in the Gaussian surface is

$$Q_{enc} = \sigma A$$

$$-E_{I} A + E_{II} A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

<u>Symmetry</u> $E_{II} = -E_{I}$

$\Rightarrow$ In general for a planar system, since the fields do not change strength with distance, the outermost fields must be equal, but opposite.

$$E_{II} A + E_{II} A = 2 E_{II} A = \frac{\sigma A}{\varepsilon_0}$$

$$E_{II} = \frac{\sigma}{2\varepsilon_0}$$

$$\vec{E}_{II} = \frac{\sigma}{2\varepsilon_0} \hat{x}$$

$$\vec{E}_{I} = -\frac{\sigma}{2\varepsilon_0} \hat{x}$$

Method II Sort the signs out yourself

The field must point away from the plane if $\sigma > 0$ so $\vec{E}_I$ points in the negative direction and $\vec{E}_{II}$ points in the positive direction.

$$\Phi_{e,l} > 0 \quad , \quad \Phi_{e,r} > 0$$

Let $\vec{E}_I = -|E_I|\hat{x}$ $\qquad \vec{E}_{II} = +|E_{II}|\hat{x}$

$$\Phi = |E_I|A + |E_{II}|A = Q_{enc}/\varepsilon_0 = \frac{\sigma A}{\varepsilon_0}$$

The magnitudes of the fields must be equal by symmetry.

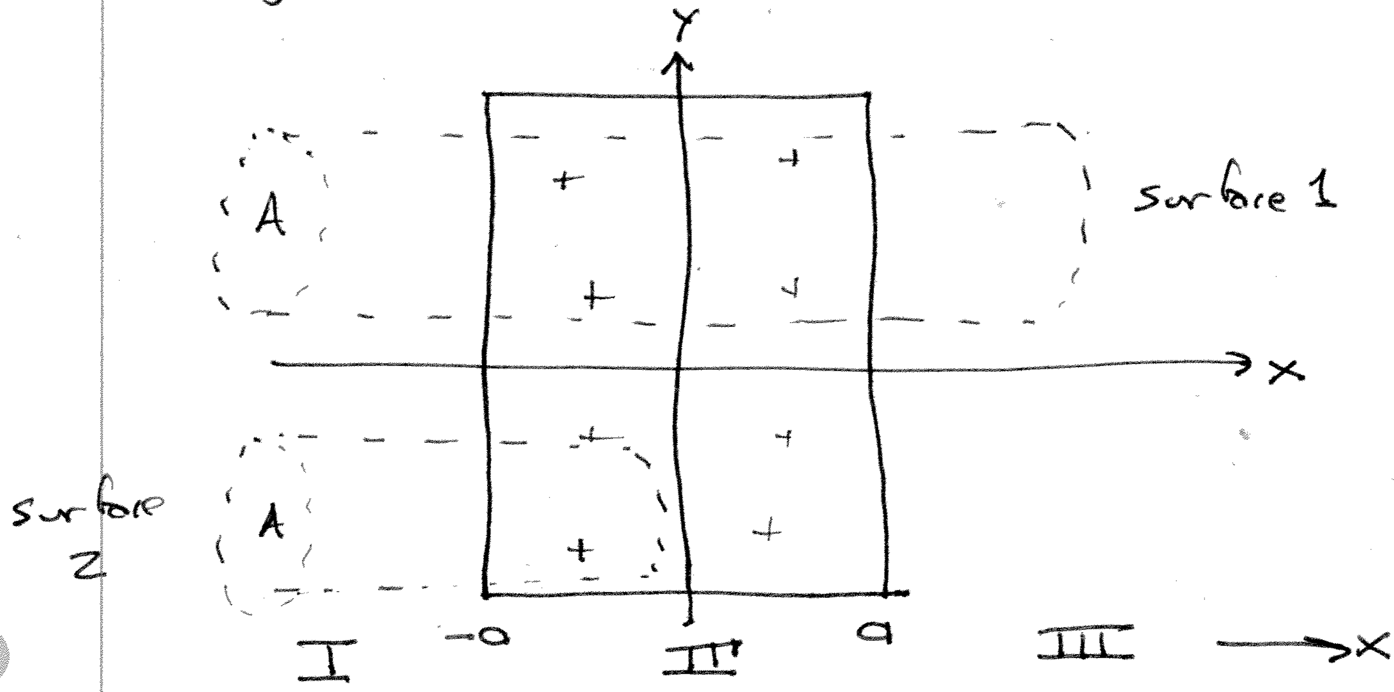
$$|E_I| = |E_{II}|$$

$$|E_{II}|A + |E_{II}|A = \frac{\sigma A}{\varepsilon_0}$$

$$|E_{II}| = |E_I| = \frac{\sigma}{2\varepsilon_0}$$

$$\vec{E}_{II} = \frac{\sigma}{2\varepsilon_0}\hat{x} \qquad \vec{E}_I = \frac{-\sigma}{2\varepsilon_0}\hat{x}$$

<u>Ex</u> An infinite planar slab of charge with volume charge density ρ = constant in region $-a < x < a$, $-\infty < y < \infty$, $-\infty < z < \infty$.



Use two Gaussian surfaces.

<u>Surface 1</u> $\quad Q_{enc} = \rho V = 2aA\rho$

$$\Phi_e = -E_I A + E_{III} A = \frac{Q_{enc}}{\varepsilon_0}$$

$$= \frac{2aA\rho}{\varepsilon_0}$$

<u>Symmetry</u> $\quad E_{III} = -E_I$

$$2E_{III} = 2a\rho/\varepsilon_0$$

$$E_{III} = a\rho/\varepsilon_0$$

$$\vec{E}_{III} = a\rho/\epsilon_0 \, \hat{x} \qquad \vec{E}_{I} = -a\rho/\epsilon_0 \, \hat{x}$$

Surface 2

$$Q_{enc} = \rho V = \rho A(x+a)$$

$$\Phi_e = -E_I A + E_{II} A = \frac{Q_{enc}}{\epsilon_0}$$

$$= \frac{\rho A(x+a)}{\epsilon_0}$$

$$E_{II}(x) = E_I + \frac{\rho}{\epsilon_0}(x+a)$$

$$= -\frac{a\rho}{\epsilon_0} + \rho/\epsilon_0 (x+a)$$

$$= \frac{\rho x}{\epsilon_0}$$

$$\vec{E} = \begin{cases} -\frac{a\rho}{\epsilon_0}\hat{x} & x < -a \\ \frac{\rho x}{\epsilon_0}\hat{x} & -a < x < a \\ \frac{a\rho}{\epsilon_0}\hat{x} & x > a \end{cases}$$

Could this possibly be correct?

I. Check symmetry

II. Check Units

$$\left[\frac{\rho x}{\epsilon_0}\right] = \frac{[\rho][x]}{[\epsilon_0]}$$

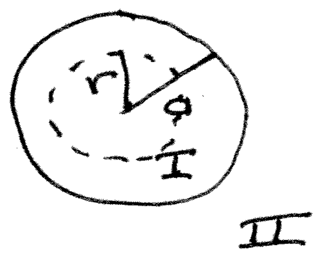
$$= \frac{(C/m^3)\cdot(m)}{C^2/Nm^2}$$

$$= \frac{C/m^2}{C^2} \cdot Nm^2$$

$$= N/C \checkmark$$

$\underline{\text{Ex}}$ Compute ~~symmetry~~ field of a system with volume charge $\rho = \gamma r$ (γ constant), for $r<a$
$\rho = 0$ for $r > a$.

$\underline{\text{S/n}}$



$\underline{\text{For } r<a}$ Only part of charge enclosed.

Use a spherical Gaussian surface, S, of radius $r<a$.

$$Q_{enc} = \int_S \rho \, d\tau$$

$$= \int_0^r dr \int_0^{\pi} r\,d\theta \int_0^{2\pi} r \sin\theta \, d\phi \, \rho$$

$$= \int_0^r 4\pi r^2 \rho(r) \, dr$$

$$= \int_0^r 4\pi r^2 \gamma r \, dr = 4\pi\gamma \int_0^r r^3 dr$$

$$Q_{enc} = 4\pi\gamma \frac{r^4}{4} = \pi\gamma r^4$$

Gauss Law

$$\Phi_e = 4\pi r^2 E_I = \frac{Q_{enc}}{\varepsilon_0}$$

$$E_I = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} = \frac{\pi\gamma r^4}{4\pi\varepsilon_0 r^2}$$

$$= \frac{\gamma r^2}{4\varepsilon_0}$$

For $r > a$ All charge enclosed

$$Q_{enc} = \int \rho \, d\tau = \int_0^a 4\pi r^2 \rho(r) \, dr$$

$$= \pi\gamma a^4$$

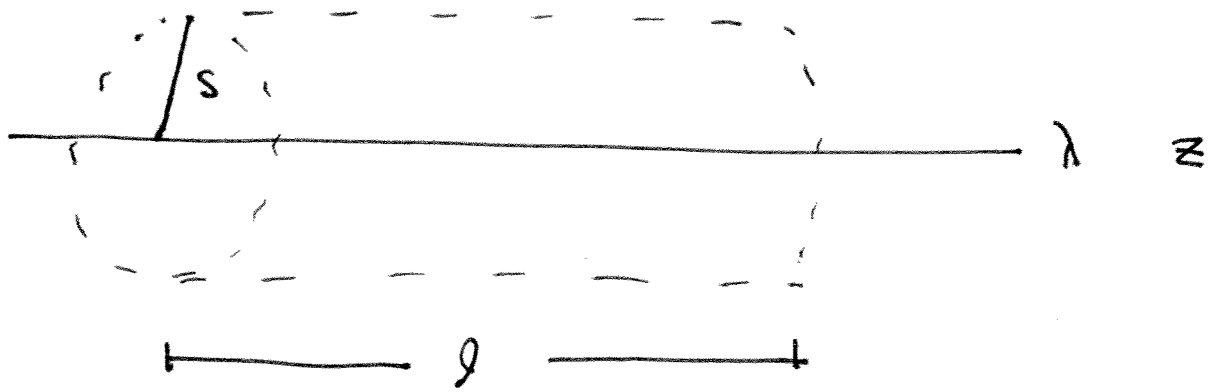
Gauss $\quad \Phi_e = 4\pi r^2 E_{II} = \frac{Q_{enc}}{\varepsilon_0}$

$$E_{II} = \frac{\pi\gamma a^4}{4\pi\varepsilon_0 r^2} = \frac{\gamma a^4}{4\varepsilon_0 r^2}$$

$$\vec{E} = \begin{cases} \dfrac{\gamma r^2}{4\varepsilon_0} & r < a \\[2ex] \dfrac{\gamma a^4}{4\varepsilon_0 r^2}\,\hat{r} & r > a \end{cases}$$

Ex Field of an infinite linear charge with charge density λ.

Sln Let the charge lie along the z-axis. By symmetry the field is in the $\hat{s}$ direction



The flux out the left and right ends of the Gaussian surface is zero.

$$\Phi_e = |E| A_s = 2\pi s l E(s) = \frac{Q_{enc}}{\varepsilon_0}$$

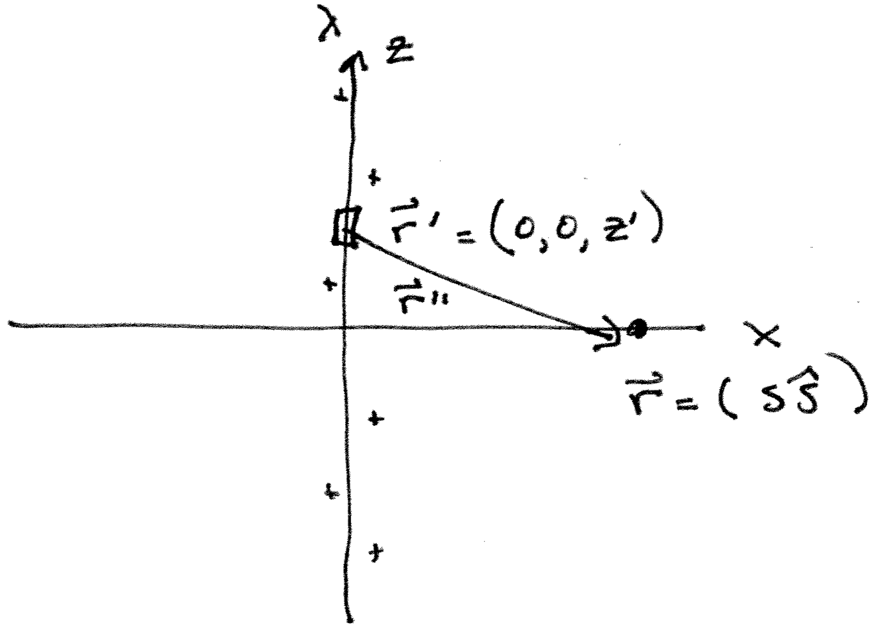
$$Q_{enc} = \lambda l$$

$$2\pi s l E(s) = \lambda l / \varepsilon_0$$

$$\vec{E}(s) = \frac{\lambda}{2\pi\varepsilon_0 s} \hat{s}$$

Let's try using Coulomb's Law

By the translational symmetry along the line, the field is ~~constant~~ the same at all z, so choose to work in the x-y plane



Source point $\vec{r}' = z'\hat{z}$

Field point $\vec{r} = s\hat{s}$

Displacement $\vec{r}'' = \vec{r} - \vec{r}' = s\hat{s} - z'\hat{z}$

$$r'' = \sqrt{s^2 + z'^2}$$

$$\hat{r}'' = \frac{s}{\sqrt{s^2+z'^2}}\hat{s} - \frac{z'}{\sqrt{s^2+z'^2}}\hat{z}$$

Coulomb's Law

$$\vec{E}(\vec{r}) = \int_{\text{line}} \frac{\lambda \, dl'}{4\pi\varepsilon_0 \, r''} \hat{r}'' \qquad dl = dz'$$

$$= \int_{-\infty}^{\infty} dz' \left(\frac{\lambda}{4\pi\varepsilon_0}\right) \frac{1}{(s^2+z'^2)} \cdot$$

$$\frac{s}{\sqrt{s^2+z'^2}} \hat{s} - \frac{z'}{\sqrt{s^2+z'^2}} \hat{z}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{s \, \hat{s} \, dz'}{(\sqrt{s^2+z'^2})^3}$$

$$- \frac{\lambda}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{z' \, \hat{z} \, dz'}{(s^2+z'^2)^{3/2}}$$

$$= \frac{\lambda s \hat{s}}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{dz'}{(s^2+z'^2)^{3/2}}$$

$$- \frac{\lambda \hat{z}}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{z' \, dz'}{(s^2+z'^2)^{3/2}}$$

The second integral is zero because it integrates an odd function over an even range.

$$\vec{E}(\vec{r}) = \frac{2\lambda s \hat{s}}{4\pi\varepsilon_0} \int_0^{\infty} \frac{dz'}{(s^2+z'^2)^{3/2}}$$

$$= \frac{\lambda s \hat{s}}{2\pi\varepsilon_0} \left[\frac{z'}{s^2\sqrt{z'^2+s^2}} \right]_0^{\infty}$$ integral table

$$= \frac{\lambda s \hat{s}}{2\pi\varepsilon_0} \left[\frac{1}{s^2} \right]$$

$$= \frac{\lambda}{2\pi\varepsilon_0 s} \hat{s}$$

Now, what about the potential?

$$V(\vec{r}) = \int \frac{\lambda \, d\ell}{4\pi\varepsilon_0 r''}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{dz'}{\sqrt{s^2+z'^2}}$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \int_0^{\infty} \frac{dz'}{\sqrt{s^2+z'^2}}$$

$$V(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(z' + \sqrt{z'^2 + s^2}\right)\Bigg|_0^{\infty}$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \sinh^{-1}\left(\frac{z'}{s}\right)\Bigg|_0^{\infty}$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \sinh^{-1}(\infty) \quad = \infty$$

$\Rightarrow$ sometimes the potential is problematic for these infinite charge distributions

$\Rightarrow$ We need to choose a different reference point