Homework 2

Due Friday 2/1/2013 - at beginning of class

Reading Assignment - Chapter 2.1-2.3

Griffiths Problems, 4th Edition

Each problem should be started on its own piece of paper. Points will be removed from solutions that are difficult to read.

When the problem numbering is different between the 3rd and 4th edition of Griffiths, the third edition number is in parenthesis.

1.44

X **2.6**

X **2.15**

2.17

2.20

2.23

X **Problem E.2.1** A non-uniform semi-infinite cylindrical volume charge with volume charge density $\rho = \gamma s^2$ occupies the region $s < a$. Compute the electric field inside ($s < a$) and the field outside ($s > a$) the volume charge, that is compute the field everywhere.

X **Problem E.2.2** For each of the following fields, determine if the field is a possible electromagnetic field; that is, which of the following fields obey Maxwell's equations.

(a) $\vec{E} = \gamma r^2 \hat{r}$ in spherical coordinates

(b) $\vec{B} = \gamma s^2 \hat{s}$ in cylindrical coordinates

Problem E.2.3 A square of charge with uniform charge density σ lies in the $x - y$ plane, centered at the origin. The square has side of length $2a$. Calculate the field at a point a distance $R > a$ along the y axis.

(1.44)

(a)

$x = 3$ is in the integral range

$$\int_2^6 (3x^2 - 2x - 1)\,\delta(x-3)\,dx = \underbrace{3(3)^2}_{27} \underbrace{- 2\cdot 3}_{-6} \underbrace{- 1}_{-1}$$

$$= 20$$

(b) $\pi \in [0, 5]$ so

$$\int_0^5 \cos x \,\delta(x - \pi)\,dx = \cos \pi = -1$$

(c) $-1 \notin [0, 3]$

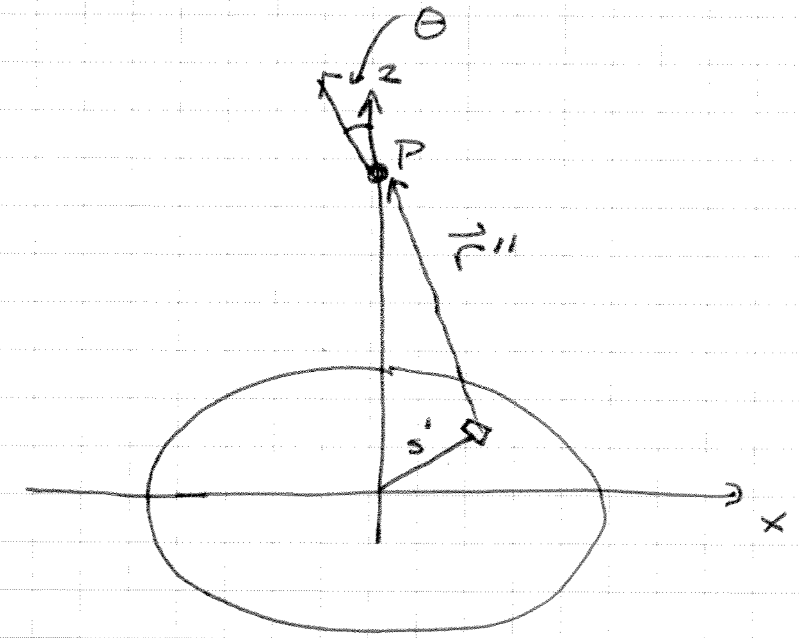
$$\int = 0$$

(d) $x = -2 \in [-\infty, \infty]$

$$\int_{-\infty}^{\infty} \ln(x+3)\,\delta(x+2)\,dx = \ln(-2+3) = \ln(1)$$

$$= 0$$

2.6



$\vec{r}_p = (0, 0, z) \qquad \vec{r}\,' = s'\hat{s}'$

$$\vec{r}'' = \vec{r}_p - \vec{r}\,' = -s'\hat{s}' + z\hat{z}$$

$$r'' = \sqrt{s'^2 + z^2}$$

The field points in the +z direction by symmetry, so only the z-component of the field survives integration.

$$|d\vec{E}|_z = \frac{k\,dq}{r''^2}\cos\Theta$$

$$\cos\Theta = \frac{z}{\sqrt{z^2+s'^2}}$$

$$|E|_z = \int \frac{k\,dq}{r''^2} \frac{z}{\sqrt{z^2+s'^2}}$$

$$= \int \frac{k\sigma\, da'\; z}{(z^2+s'^2)^{3/2}}$$

$$= k\sigma z \int_0^R ds' \int_0^{2\pi} s'\,d\phi' \frac{1}{(z^2+s'^2)^{3/2}}$$

$$= 2\pi k\sigma z \int_0^R ds' \frac{1}{(z^2+s'^2)^{3/2}}$$

$$= 2\pi k\sigma z \left(\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}} \right)$$

$$\vec{E} = \frac{\sigma z \hat{z}}{2\varepsilon_0} \left(\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}} \right)$$

- If $R \to \infty$, $\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{z}$ the field of an infinite plane.

- If $z \to \infty$, $\frac{1}{\sqrt{R^2+z^2}} \to \frac{1}{z\sqrt{1+R^2/z^2}}$

$$= \frac{1}{z}\left(1 - \frac{1}{2}\frac{R^2}{z^2} + \cdots\right)$$

by the binomial thm $(1+x)^n \sim 1 + nx + \cdots$

$$\vec{E} \rightarrow \frac{\sigma z \hat{z}}{2\varepsilon_0}\left(\frac{1}{z} - \frac{1}{z}\left(1 - \frac{1}{2}\frac{R^2}{z^2}\right)\right)$$

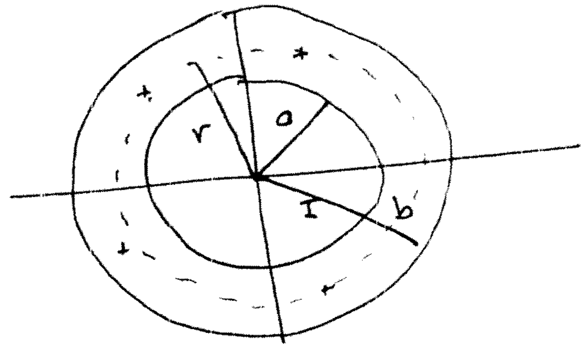
$$= \frac{\sigma R^2}{4\varepsilon_0 z^2}\hat{z} = \frac{\cancel{4}(\pi R^2 \sigma)}{4\pi\varepsilon_0 z^2}\hat{z}$$

= Field of point charge with charge

$Q = \pi R^2 \sigma$

2.15

$$\rho = \frac{k}{r^2}$$



The charge enclosed in a Gaussian surface with $a < r < b$ drawn above

$$Q_{enc} = \int \rho \, d\tau$$

$$= \int_a^{b} 4\pi r^2 \rho \, dr$$

$$= \int_a^r 4\pi r^2 \cdot \frac{k}{r^2} \, dr$$] shell method

$$= 4\pi k (r - a)$$

The total charge of the shell is

$$Q_T = 4\pi k (b - a)$$

Inside all charge $\quad r < a$

$$\vec{E} = 0$$

Outside all charge $\quad r > b$

$$\vec{E} = \frac{Q_T}{4\pi\varepsilon_0 r^2}\hat{r}$$

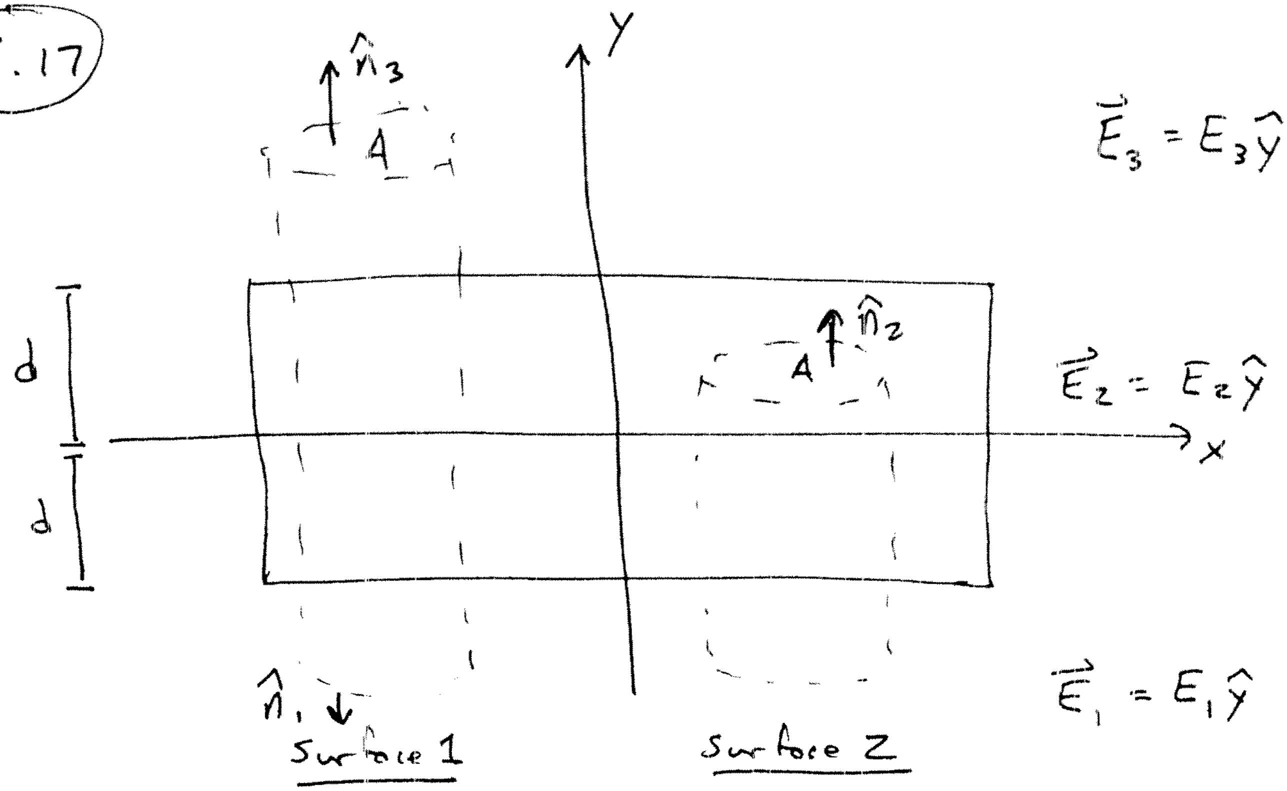
field of point charge with total charge of system.

In volume charge $\quad a < r < b$

$$\Phi = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0} \quad \text{Gauss Law}$$

$$\vec{E} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{4\pi k(r-a)}{4\pi\varepsilon_0 r^2}\hat{r}$$

2.17



Surface 1 $\quad Q_{enc} = 2dA\rho \qquad \hat{n}_3 = \hat{y} \qquad \hat{n}_1 = -\hat{y}$

$$\Phi = \vec{E}_3 \cdot \hat{n}_3 A + \vec{E}_1 \cdot \hat{n}_1 A$$

$$= E_3 A + (-E_1 A) = \frac{Q_{enc}}{\varepsilon_0} \qquad \text{Gauss}$$

By symmetry $E_3 = -E_1$

$$2E_3 A = \frac{Q_{enc}}{\varepsilon_0} = \frac{2dA\rho}{\varepsilon_0}$$

$$E_3 = \frac{d\cancel{A}\rho}{\varepsilon_0} = \frac{d\rho}{\varepsilon_0}$$

$$\vec{E}_3 = \frac{\rho d}{\varepsilon_0}\hat{y} \qquad \vec{E}_1 = -\frac{\rho d}{\varepsilon_0}\hat{y}$$

<u>Surface 2</u> $\quad Q_{enc} = \rho A(y+d)$

$$\Phi = \vec{E}_2 \cdot \hat{n}_2 A + \vec{E}_1 \cdot \hat{n}_1 A$$

$$= E_2 A - E_1 A = \frac{Q_{enc}}{\varepsilon_0} = \rho\frac{A(y+d)}{\varepsilon_0}$$

$$E_2 = E_1 + \frac{\rho(y+d)}{\varepsilon_0}$$

$$= -\frac{\rho d}{\varepsilon_0} + \frac{\rho(y+d)}{\varepsilon_0} = \frac{\rho y}{\varepsilon_0}$$

$$\vec{E}_2 = \frac{\rho y}{\varepsilon_0}\hat{y}$$

2.20 To be possible, the field must be curl free.

(a)

$$\nabla \times \vec{E} = k \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix}$$

$$= \hat{x}(0 - 2y) \neq 0$$

$$+ \hat{y}(\;)$$

$$+ \hat{z}(\;)$$

not an electric field

(b)

$$\nabla \times \vec{E} = k \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & (2xy + z^2) & 2yz \end{vmatrix}$$

$$= \left[\hat{x}(2z - 2z) - \hat{y}(0 - 0) + \hat{z}(2y - 2y)\right] k$$

$= 0$ can be electric field.

$$\vec{E} = -\nabla V$$

$$\vec{E} \cdot d\vec{l} = -\frac{\partial V}{\partial x} dx - \frac{\partial V}{\partial y} dy - \frac{\partial V}{\partial z} dz$$

$$\frac{\partial V}{\partial x} = -ky^2$$

$$V = -kxy^2 + f(y,z)$$ where f is some function.

$$\frac{\partial V}{\partial y} = -k(2xy + z^2) = -2kxy + \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} = -kz^2$$

$$f = -kyz^2 + g(z)$$

$$V = -kxy^2 - kyz^2 + g(z)$$

$$\frac{\partial V}{\partial z} = -2kyz = -2kyz + \frac{\partial g}{\partial z}$$

$g =$ constant

$$V = -kxy^2 - kyz^2 + C$$

$$\vec{E} = -\nabla V = \left(+ky^2,\ 2kxy + kz^2,\ 2kyz \right) \checkmark$$

2.28

$$\vec{E} = \begin{cases} 0 & r<a \\ \dfrac{k(r-a)}{r^2\varepsilon_0}\,\hat{r} & a<r<b \\ \dfrac{Q_T}{4\pi\varepsilon_0 r^2}\,\hat{r} & r>b \end{cases}$$

$$V = -\int E\,dr$$

$$V = \frac{Q_T}{4\pi\varepsilon_0 r} + C_{III} \quad (r>b)$$, point charge field

$$V = C_I \qquad r<a$$, since $E=0$

$$V = -\frac{k}{\varepsilon_0}\int\left(\frac{1}{r} - \frac{a}{r^2}\right)dr \qquad a<r<b$$

$$= -\frac{k}{\varepsilon_0}\left(\ln(r) + \frac{a}{r}\right) + C_{II}$$

where C_I, C_{III}, C_{II} are constants of integration.

The potential must be continuous.

Select C_{III} so $V(\infty) = 0 \Rightarrow C_{III} = 0$

$$V(b) = -\frac{k}{\varepsilon_0}\left(\ln(b) + \frac{a}{b}\right) + C_{II} = \frac{Q_T}{4\pi\varepsilon_0 b}$$

$$C_{II} = \frac{Q_T}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\ln(b) + \frac{k}{\varepsilon_0}\frac{a}{b}$$

$$V(r) = \frac{Q_T}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\left(\ln(b) - \ln(r)\right) + \frac{ka}{\varepsilon_0}\left(\frac{1}{b} - \frac{1}{r}\right)$$

$$= \frac{Q_T}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\ln\left(\frac{b}{r}\right) + \frac{ka}{\varepsilon_0 br}(r-b)$$

$a < r < b$

Match potential at $r = a$

$$V(a) = C_I = \frac{Q_T}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\ln\left(\frac{b}{a}\right) + \frac{ka}{\varepsilon_0 ba}(a-b)$$

$$Q_T = 4\pi k (b-a)$$

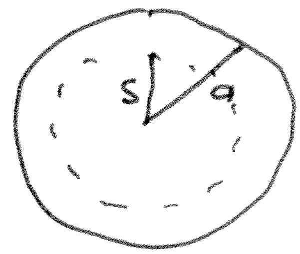
$$V(a) = \frac{4\pi k(b-a)}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\ln\left(\frac{b}{a}\right) + \frac{k}{\varepsilon_0 b}(a-b)$$

$$= \frac{k}{\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

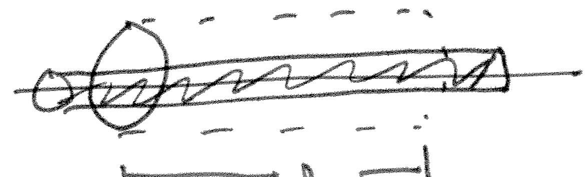
E.2.1 A non-uniform cylindrical volume charge occupies the region $s<a$. The volume charge density varies as $\rho(s) = \gamma s^2$. Compute the field everywhere.

Inside Charge $(s<a)$

$$Q_{enc} = \int_{\substack{\text{Gaussian}\\ \text{surface}}} \rho \, d\tau$$



$$= l \int \rho \, da$$



$$da = ds \, s \, d\phi$$

$$Q_{enc} = l \int_0^s ds \int_0^{2\pi} s \, d\phi \, \rho$$

$$= l \int_0^s ds \int_0^{2\pi} \gamma s^2 d\phi$$

$$= 2\pi \gamma l \int_0^s s^3 ds = 2\pi\gamma l \frac{s^4}{4}$$

$$Q_{enc} = \frac{\pi \gamma l s^4}{2}$$

For the cylindrical surface, the flux out of the surface is

$$\Phi_e = 2\pi s l E = Q_{enc}/\varepsilon_0 \quad \text{(Gauss)}$$

$$E = \frac{Q_{enc}}{2\pi \varepsilon_0 s l}$$

$$= \frac{\pi \gamma l s^4 / 2}{2\pi s l \varepsilon_0}$$

$$= \frac{\gamma s^3}{4 \varepsilon_0}$$

$$\vec{E} = \frac{\gamma s^3}{4\varepsilon_0} \hat{s}$$

<u>For $s > a$</u> (outside sphere)

$$Q_{enc} = \frac{\pi \gamma l a^4}{4}$$

$$\vec{E} = \frac{Q_{enc}}{2\pi \varepsilon_0 l s} \hat{s} = \frac{\pi \gamma l a^4 / \cancel{2}}{2\pi \varepsilon_0 l s} \hat{s} = \frac{\gamma a^4}{\cancel{4}\varepsilon_0 s} \hat{s}$$

E.2.2

(a) $\vec{E} = \gamma r^2 \hat{r}$

To be a possible EM field, the field must satisfy Maxwell's Eqns for some, ρ, $\vec{J}$, $\vec{B}$.

Gauss $\nabla \cdot \vec{E} = \rho/\varepsilon_0$

$$= \frac{1}{r^2} \frac{\partial r^2 A_r}{\partial r}$$

$$= \frac{\gamma}{r^2} \frac{\partial r^4}{\partial r} = \frac{4\gamma r^3}{r^2}$$

$$= 4\gamma r$$

Gauss' Law is satified if $\rho = 4\gamma r$.

Faraday $\nabla \times \vec{E} = 0$ satified if $\vec{B} = 0$

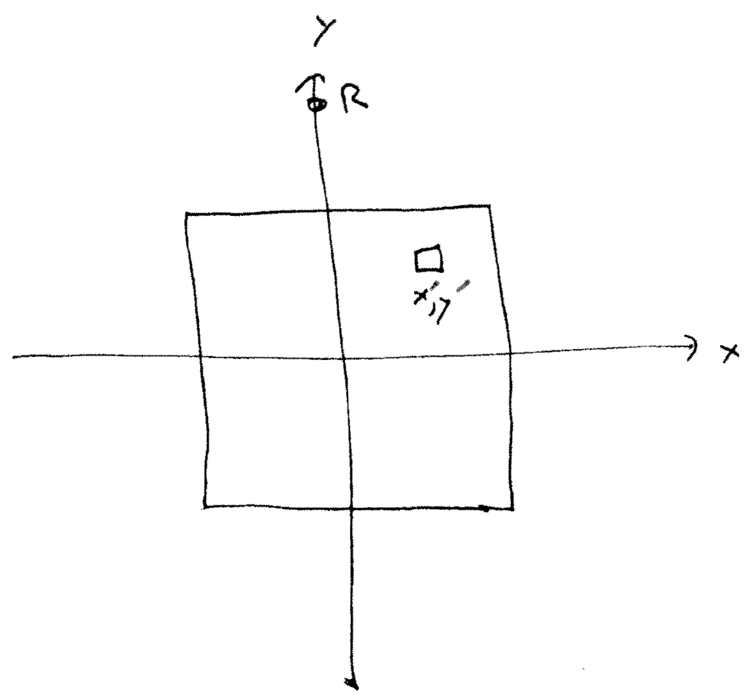
No Magnetic monopoles and Ampere satified if $\vec{B} = 0$, $\vec{J} = 0$, and $\vec{E}$ constant ✓

This is a possible field.

(b) $\nabla \cdot \vec{B} \neq 0 \implies$ Not a possible EM field.

E.2.3

~~Prb 4~~ A square of charge $2a$ on side with surface charge density σ.



$\vec{r} = (0, R, 0)$ $\vec{r}\,' = (x', y', 0)$

$\vec{r}\,'' = \vec{r} - \vec{r}\,' = (-x', R-y', 0)$

$r'' = \sqrt{x'^2 + (R-y')^2}$

$$\vec{E} = \int \frac{k\sigma\, dx'\, dy'\, \vec{r}\,''}{r''^3}$$

By symmetry the field is in y direction, take y component

$$\vec{E} = \hat{y}\, k\sigma \int_{-a}^{a}\int_{-a}^{a} \frac{(R-y')\,dx'\,dy'}{\left(x'^2+(R-y')^2\right)^{3/2}}$$

$$u = R-y' \qquad du = -dy'$$

$$\vec{E} = -\hat{y}\, k\sigma \int_{-a}^{a} dx' \int_{R+a}^{R-a} \frac{u\,du}{\left(x'^2+u^2\right)^{3/2}}$$

$$= -\hat{y}\, k\sigma \int_{-a}^{a} dx' \left[-\frac{1}{\sqrt{u^2+x^2}} \right]_{R+a}^{R-a}$$

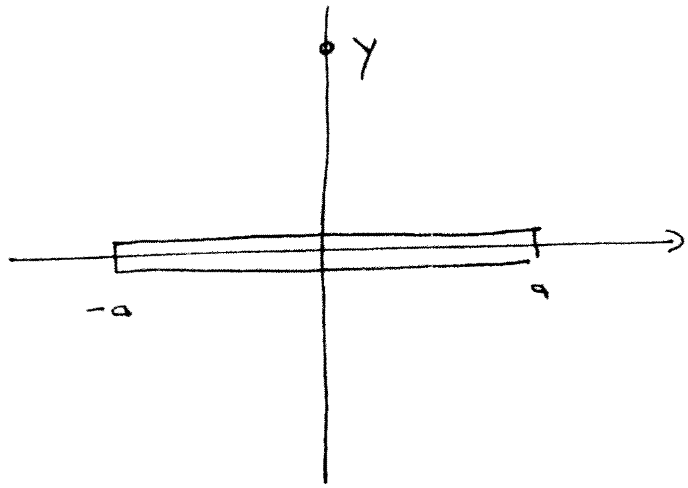
$$= -\hat{y}\, k\sigma \int_{-a}^{a} dx' \left[\frac{1}{\sqrt{(R+a)^2+x'^2}} - \frac{1}{\sqrt{(R-a)^2+y'^2}} \right]$$

$$= -\hat{y}\, 2k\sigma \int_{0}^{a} dx' \left[\frac{1}{\sqrt{(R+a)^2+x'^2}} - \frac{1}{\sqrt{(R-a)^2+y'^2}} \right]$$

$$= -\hat{y}\, 2k\sigma \left[\ln\left(x+\sqrt{x^2+(R+a)^2}\right) - \ln\left(x+\sqrt{x^2+(R-a)^2}\right) \right]_{0}^{a}$$

$$= 2\sigma k\hat{y} \left[\ln\left(\frac{a+\sqrt{a^2+(R-a)^2}}{|R-a|} \right) - \ln\left(\frac{a+\sqrt{a^2+(R+a)^2}}{|R+a|} \right) \right]$$

Method II - Compute field of line charge first.



$\vec{r}' = (x', 0, 0) \qquad \vec{r}(0, y, 0)$

$\vec{r}'' = (-x', y, 0)$

$$\vec{E} = \lambda k \int_{-a}^{a} \frac{dx' \, \vec{r}''}{r''^3}$$

$$E_y = \lambda k y \int_{-a}^{a} \frac{dx'}{\left(\sqrt{x'^2 + y^2}\right)^3}$$

$$= 2\lambda k y \left. \frac{x'}{y^2\sqrt{x'^2+y^2}} \right|_0^a$$

$$= \frac{2\lambda k y a}{y^2\sqrt{y^2+a^2}} = \frac{2\lambda k}{y}\left(\frac{a}{\sqrt{y^2+a^2}}\right)$$

Correct as $a \to \infty$

Now build square out of sequence of strips

$$\vec{E}_{square} = \int_{-a}^{a} \vec{E}_{strip}(R-y')\,dy' \qquad \lambda = \sigma\,dy'$$

$$\vec{E}_{square} = 2\sigma k \hat{y}\, a \int_{-a}^{a} \frac{dy'}{(R-y')\sqrt{(R-y')^2+a^2}}$$

$$u = R-y' \qquad du = \cancel{R} = -dy'$$

$$\vec{E}_{square} = -2\sigma k a \hat{y} \int_{R+a}^{R-a} \frac{du}{u\sqrt{u^2+a^2}}$$

$$= -2k\sigma a \hat{y}\left(-\frac{1}{a}\ln\left(\frac{a+\sqrt{u^2+a^2}}{u}\right)\right)\Bigg|_{R+a}^{R-a}$$

$$= -2k\sigma \hat{y}\left(\ln\left(\frac{a+\sqrt{(R-a)^2+a^2}}{R-a}\right)\right.$$

$$\left. - \ln\left(\frac{a+\sqrt{(R+a)^2+a^2}}{R+a}\right)\right)$$