Homework 3

Due Friday 2/8/2013 - at beginning of class

Reading Assignment - Chapter 2.4-2.5

Griffiths Problems, 4th Edition

Each problem should be started on its own piece of paper. Points will be removed from solutions that are difficult to read.

When the problem numbering is different between the 3rd and 4th edition of Griffiths, the third edition number is in parenthesis.

2.30 Parts (b) and (c) only

2.31

2.41 (Griffiths 3rd Edition problem 2.37)

2.42 (Griffiths 3rd Edition problem 2.38)

2.43 (Griffiths 3rd Edition problem 2.39)

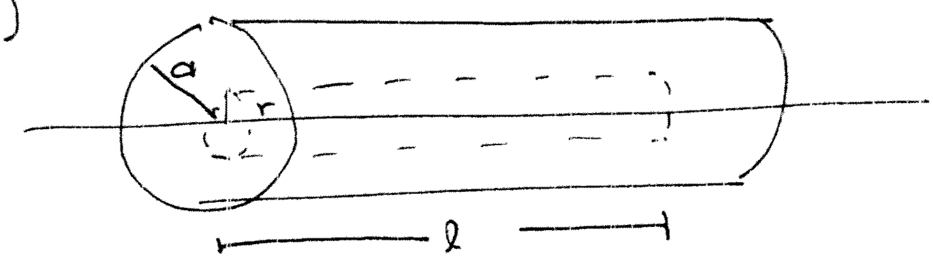
2.48 (Griffiths 3rd Edition problem 2.44)

2.50 (Griffiths 3rd Edition problem 2.46) Find $\vec{E}$ only.

Problem E.3.1 A spherical region $r < a$ contains a non-uniform volume charge $\Gamma = \gamma r^3$ where γ is a constant. Compute the field everywhere.
Problem E.3.2 A ring with radius a and constant linear charge density λ lies in the $x - y$ plane centered at the origin. Compute the electric potential at a point R along the z-axis.
Problem E.3.3 A ring with radius a and constant linear charge density λ lies in the $x - y$ plane centered at the origin. Compute the electric field at a point R along the z-axis.
Problem E.3.4 Two spherical shells of radius a and b, $a < b$, have uniformly distributed charges $Q_a = Q$ and $Q_b = -Q$. Compute the energy between the shells.

2.30

(b)



Cylindrical Gaussian Surface radius r and length l.

$$\Phi = 2\pi r l E = \frac{Q_{enc}}{\varepsilon_0}$$

For $r < a$, $Q_{enc} = 0$, $E = 0$.

For $r > a$, $Q_{enc} = 2\pi a l \sigma$

$$E = \frac{Q_{enc}}{2\pi r l \varepsilon_0} = \frac{2\pi a l \sigma}{2\pi r l \varepsilon_0}$$

$$= \frac{a\sigma}{r\varepsilon_0}$$

B.C. $E_2 - E_1 = \frac{\sigma}{\varepsilon_0} = \frac{a\sigma}{a\varepsilon_0} - 0 = \frac{\sigma}{\varepsilon_0}$ ✓

(c) From example 2.7

$$V_{out}(z) = \frac{R^2\sigma}{\epsilon_0 z} \qquad \text{outside}$$

$$V_{in}(z) = \frac{R\sigma}{\epsilon_0} \qquad \text{inside}$$

2.34 Continuous

$$V_{out}(R) = \frac{R^2\sigma}{\epsilon_0 R} = \frac{R\sigma}{\epsilon_0} = V_{in}(R) = \frac{R\sigma}{\epsilon_0}$$

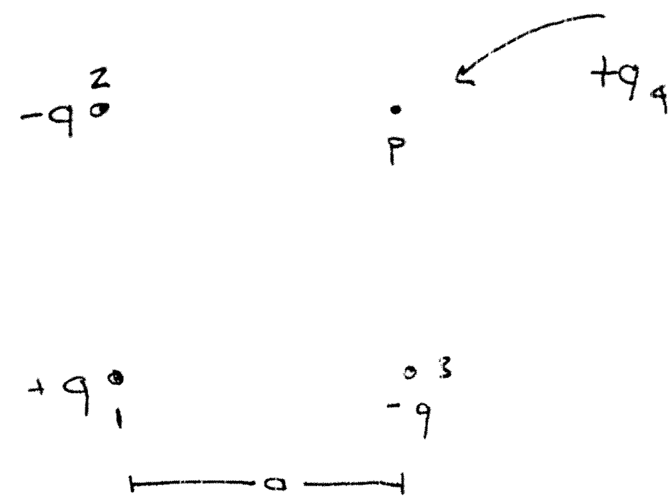
2.36

$$\left.\frac{\partial V_{out}}{\partial z}\right|_R - \left.\frac{\partial V}{\partial z}\right|_{in} = \frac{-\sigma}{\epsilon_0}$$

$$-\left.\frac{R^2\sigma}{\epsilon_0 z^2}\right|_R - 0 = -\frac{\sigma}{\epsilon_0} = \frac{-\sigma}{\epsilon_0} \checkmark$$

(2.31)

(a)



$$W = q V_p$$

$$= q\left(\frac{-kq}{a} + \frac{-kq}{a} + \frac{kq}{\sqrt{2}\,a} \right)$$

$$= \frac{kq^2}{a}\left(\frac{1}{\sqrt{2}} - 2 \right)$$

(b) The work to build the system is

$$W_1 = 0$$

$$W_2 = \frac{-kq^2}{a}$$

$$W_3 = -q\left(\frac{kq}{a} + \frac{-kq}{\sqrt{2}\,a} \right) = \frac{kq^2}{a}\left(\frac{1}{\sqrt{2}} - 1 \right)$$

$$W_4 = q\left(\frac{-kq}{a} + \frac{-kq}{a} + \frac{kq}{\sqrt{2}\,a} \right)$$

$$= \frac{kq^2}{a}\left(\frac{1}{\sqrt{2}} - 2 \right)$$

$$W = W_1 + W_2 + W_3 + W_4$$

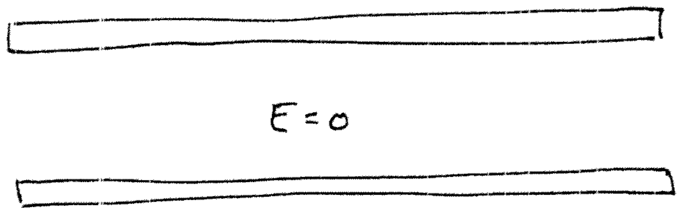
$$= 0 + \frac{kq^2}{a}\left(-1 + \frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}} - 2 \right)$$

$$= \frac{kq^2}{a}\left(\frac{2}{\sqrt{2}} - 4 \right) = \frac{2kq^2}{a}\left(\frac{1}{\sqrt{2}} - 2 \right)$$

~~2.37~~

2.41

$$E = \frac{\sigma}{\varepsilon_0}$$



$\sigma = \frac{Q}{A}$

$E = 0$

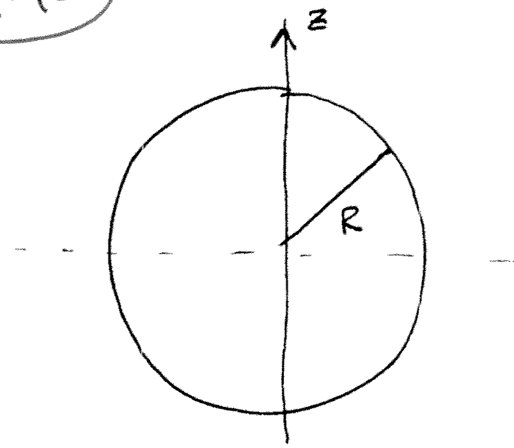
$\sigma = Q/A$

$$E = \frac{\sigma}{\varepsilon_0}$$

$$P = \sigma \frac{E}{2} = \frac{\sigma \cdot \sigma/\varepsilon_0}{2}$$

$$= \frac{\sigma^2}{2\varepsilon_0} = \frac{Q^2}{2A^2\varepsilon_0}$$

~~2.38~~ 2.42



Electric field $\qquad \vec{E} = 0 \qquad r < R$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \qquad r > R$$

Electric Pressure

$$\vec{P} = \sigma \vec{E}_{ave} = \frac{\sigma}{2}\vec{E} \qquad r > R$$

$$= \frac{\sigma Q}{8\pi\epsilon_0 R^2}\hat{r} \qquad \sigma = \frac{Q}{4\pi R^2}$$

Total force exerted on northern hemisphere

$$\vec{F} = \int \vec{P}\, da$$

$$da = R\,d\theta \sin\theta\, R\,d\phi = R^2 \sin\theta\, d\theta\, d\phi$$

Evidently only $\hat{z}$ component survives,

$$\hat{r} = \underbrace{\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y}}_{\text{integrates to zero}} + \cos\theta\,\hat{z}$$

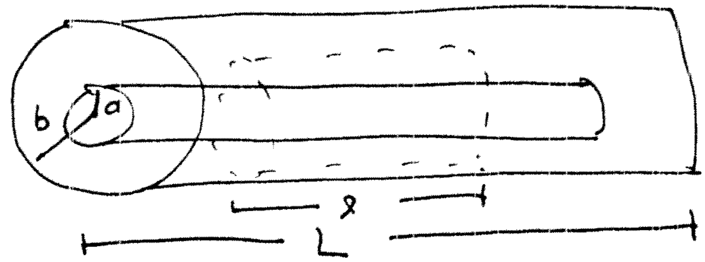
$$\vec{F} = \int \vec{P}\,da = \int \left(\frac{\sigma Q}{4\pi\varepsilon_0 R^2}\cdot\cos\theta\,\hat{z}\right)\left(R^2 \sin\theta\, d\theta\, d\phi\right)$$

$$= \frac{\sigma Q}{8\pi\varepsilon_0} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi/2} d\theta \cos\theta \sin\theta}_{\frac{1}{2}}\ \hat{z}$$

$$= \frac{\sigma Q\,\pi}{8\pi\varepsilon_0}\hat{z} = \frac{\sigma Q}{8\varepsilon_0}\hat{z}$$

$$= \frac{Q}{8\varepsilon_0}\left(\frac{Q}{4\pi R^2}\right) = \frac{Q^2}{32\pi\varepsilon_0 R^2}\hat{z}$$

~~2.39~~ 2.43



Add a charge Q to inner conductor, and charge $-Q$ to outer conductor.

Use a cylindrical Gaussian surface that encloses the inner conductor.

$$Q_{enc} = \lambda l$$

where $\lambda = Q/L$

<u>Gauss Law</u>

$$\Phi = 2\pi r l E = \frac{Q_{enc}}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi r \varepsilon_0}$$

Potential Difference

$$V = -\int_a^b E\,dr$$

$$= \frac{-\lambda}{2\pi\varepsilon_0}\ln\left(b/a\right) = -\frac{Q}{2\pi\varepsilon_0 L}\ln\left(b/a\right)$$

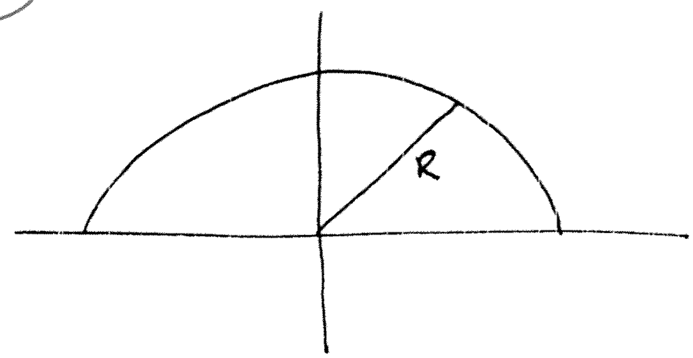
Capacitance

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{\frac{Q}{2\pi\varepsilon_0 L}\ln\left(b/a\right)}$$

$$= \left(\frac{2\pi\varepsilon_0}{\ln(b/a)}\right) L$$

Capacitance per unit length

~~2.44~~ 2.48



Center

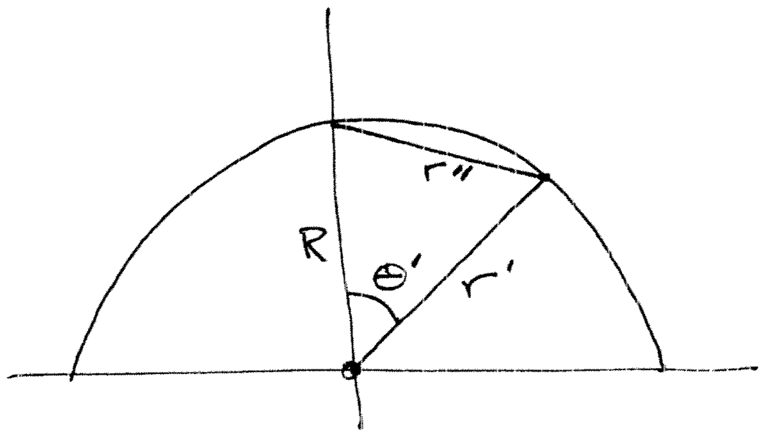
$$V(0) = \int \frac{k\sigma\,da}{R}$$

$$= \frac{k\sigma}{R} \cdot \text{Surface area}$$

$$= \frac{k\sigma}{R} \cdot 2\pi R^2 = 2\pi k\sigma R$$

$$= \frac{\sigma R}{2\varepsilon_0}$$

Pole



$$V_{pole} = \int \frac{k\sigma da'}{r'} \qquad r' = R\hat{r}'$$

Low of Cosines

$$r''^2 = R^2 + R^2 - 2R^2\cos\theta$$

$$= 2R^2(1-\cos\theta')$$

$$V_{pole} = \frac{k\sigma}{\sqrt{2}R}\int_0^{2\pi} d\phi' \int_0^{\pi/2} d\theta' \frac{R^2 \sin\theta'}{\sqrt{1-\cos\theta'}}$$

$$= \frac{2\pi k\sigma R}{\sqrt{2}} \underbrace{\int_0^{\pi/2} d\theta' \frac{\sin\theta'}{\sqrt{1-\cos\theta'}}}_{2} \quad \text{Maple}$$

$$V_{pole} = \frac{4\pi k \sigma R}{\sqrt{2}} = \frac{\sigma R}{\sqrt{2}\,\varepsilon_0}$$

$$V_{pole} - V_{center} = \frac{\sigma R}{\sqrt{2}\,\varepsilon_0} - \frac{\sigma R}{2\varepsilon_0}$$

~~2.46~~ 2.50

(a) $\vec{E} = -\nabla V = -\nabla\left(A\frac{e^{-\lambda r}}{r}\right)$

$$= \frac{A\lambda e^{-\lambda r}}{r}\hat{r} - Ae^{-\lambda r}\nabla\left(\frac{1}{r}\right)$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$$ pg 50

$$\vec{E} = \frac{A\lambda}{r}e^{-\lambda r}\hat{r} + \frac{Ae^{-\lambda r}}{r^2}\hat{r}$$

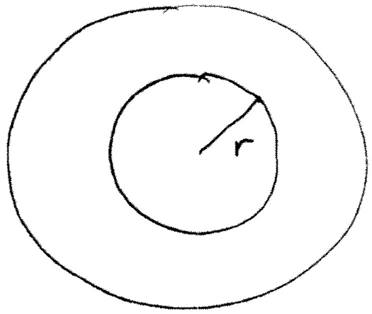
$$= \frac{Ae^{-\lambda r}}{r^2}(1+\lambda r)\hat{r}$$

(b) Charge density

$$\nabla\cdot\vec{E} = \rho/\epsilon_0$$

E.3.1

Spherical volume $\Gamma = \dfrac{\gamma}{r^2}$ $\quad r<a$



Use a spherical Gaussian surface of radius r.

If $r<a$, then the charge enclosed is

$$Q_{enc} = \int_0^r 4\pi r^2 dr\, \Gamma \; \cancel{= 4\pi\Gamma \int_0^r \frac{dr}{r^2}}$$

$$= \int_0^r 4\pi r^5 dr\, \gamma$$

$$= 4\pi\gamma \frac{r^6}{6}$$

The flux out of the Gaussian surface is

$$\Phi = EA = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0} \quad \text{by Gauss' Law}$$

$$\vec{E} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{4\pi\gamma r^6/6}{4\pi\varepsilon_0 r^2}\hat{r} \qquad r<a$$

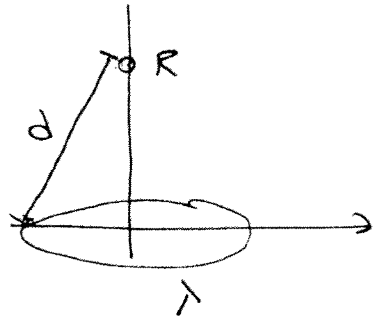
$$= \frac{r^4\gamma}{6\varepsilon_0}\hat{r} \qquad r<a$$

If $r > a$, $Q_{enc} = 4\pi\gamma a^6/6$

$$\vec{E} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{4\pi\gamma a^6/6}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$= \frac{\gamma a^6}{6\varepsilon_0 r^2}\hat{r}$$

E.3.2



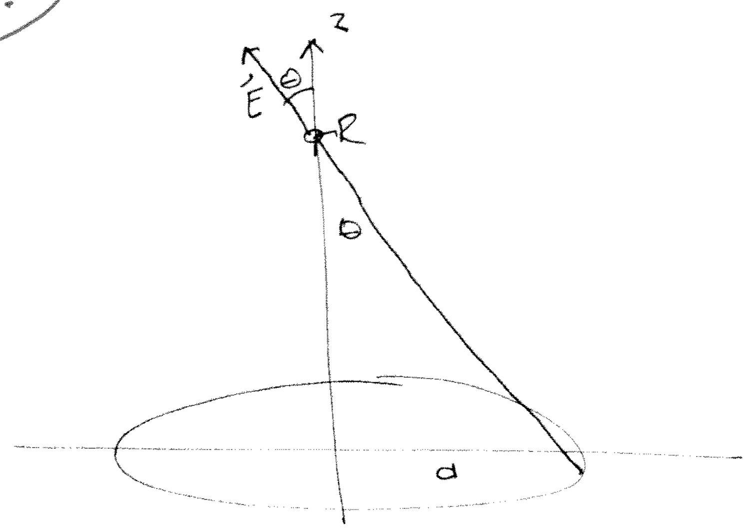
All the charge is equidistant from the loop,

$$V = \frac{kq}{d}$$

$$d = \sqrt{R^2+a^2} \qquad q = 2\pi a \lambda$$

$$V = \frac{2k\pi a \lambda}{\sqrt{R^2+a^2}} = \frac{a\lambda}{2\varepsilon_0 \sqrt{R^2+a^2}}$$

E.3.3



The field of a small chunk dq is

$$|d\vec{E}| = \frac{k\,dq}{R^2+a^2}$$

The z-components add. All other components cancel.

The z-component is

$$dE_z = \frac{k\,dq}{R^2+a^2}\cos\theta = \frac{k\,dq}{R^2+a^2}\,\frac{R}{\sqrt{R^2+a^2}}$$

The total field is the integral of this field

$$|\vec{E}| = \int \frac{Rk\,dq}{(R^2+a^2)^{3/2}} = \frac{kRQ}{(R^2+a^2)^{3/2}}$$

where $Q = 2\pi a\lambda$, the total charge of the ring.

$$\vec{E} = \frac{KR\,(2\pi a\lambda)}{(R^2+a^2)^{3/2}}\,\hat{z} = \frac{2\pi a\lambda KR}{(R^2+a^2)^{3/2}}$$

$$= \frac{Ra\lambda}{2\varepsilon_0\,(R+a^2)^{3/2}}\,\hat{z}$$

E.3.4

The electric field between the shells is

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{r}$$

The energy density is then

$$u = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2$$

$$= \frac{Q^2}{32\pi^2\varepsilon_0 r^4}$$

Integrate the energy density over the volume between the spheres.

$$U = \int u\,dv = \int_a^b dr \int_0^{2\pi} \cancel{r\,d\theta}\; r\sin\theta\, d\phi \int_0^{\pi} r\,d\theta\; u$$

$$= 4\pi\int_a^b r^2\,dr\,u = 4\pi\left(\frac{Q^2}{32\pi^2\varepsilon_0}\right)\int_a^b \frac{dr}{r^2}$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left(-\frac{1}{r}\right)\Bigg|_a^b = \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

Could also get result from capacitance.