

Homework 4

Due Friday 2/22/2013 - at beginning of class

Griffiths' 4 Problems

- 3.13 (Griffiths 3rd Edition 3.12)
- 3.19 (Griffiths 3rd Edition 3.18)
- 3.23 (Griffiths 3rd Edition 3.22) Work only up through P_3 not P_5 .

Additional Problems

E.4.1 An infinite conducting cylinder of radius a is in an external electric field that is a uniform $E_0\hat{x}$ far from the cylinder. Compute the surface charge density as a function of ϕ on the surface of the cylinder.

E.4.2 The potential at the surface of an infinite cylinder of radius a is $V(a, \phi, z) = V_0 \cos(3\phi)$. Find the potential both inside and outside the cylinder. Find the field inside and outside and the surface charge density on the cylinder.

Y eqn

$$\frac{d^2 Y}{dy^2} + k^2 Y = 0$$

$$Y = \sin kx, \cos kx$$

For BC, $V(x, 0, z) = 0$ $V(x, a, z) = 0$

chose sine solution and let $k_n = \frac{n\pi}{a}$.

X eqn

$$\frac{d^2 X}{dx^2} - k^2 X = 0$$

$$X = e^{kx}, e^{-kx}$$

For the solution to be finite as $x \rightarrow \infty$,

$$X(x) = e^{-k_n x}$$

General Solution

$$V(x, y) = \sum_n A_n e^{-k_n x} \sin k_n y$$

Impose $x=0$ boundary condition

$$V(0, y) = \begin{cases} V_0 & 0 < y < a/2 \\ -V_0 & a/2 < y < a \end{cases}$$

$$= \sum_n A_n \sin k_n y$$

Multiply by $\sin k_m y$

$$V_0 \int_0^{a/2} \sin k_m y dy - V_0 \int_{a/2}^a \sin k_m y dy$$

$$= \sum_n A_n \underbrace{\int_0^a \sin k_m y \sin k_n y dy}_{\frac{a}{2} \delta_{nm}}$$

$$\begin{aligned}
 & - \frac{V_0}{K_m} \cos k_m y \Big|_0^{a/2} + \frac{V_0}{K_m} \cos k_m y \Big|_{a/2}^a \\
 & = \frac{V_0}{2} A_m
 \end{aligned}$$

$$= - \frac{V_0 a}{m\pi} \left(\cos \frac{m\pi}{2} - 1 \right) + \frac{V_0 a}{m\pi} \left(\cos(m\pi) - \cos \frac{m\pi}{2} \right)$$

$$= \frac{V_0 a}{m\pi} \left(\cos m\pi - 2 \cos \frac{m\pi}{2} + 1 \right)$$

$$= 4 \frac{V_0 a}{m\pi} \quad \text{if } m = 2, 6, 10 \quad \text{which I simply read from the solution}$$

○ otherwise

Let's work on it a bit

$$\cos \frac{m\pi}{2} = \pm \sqrt{\frac{1 + \cos m\pi}{2}}$$

No luck, try a table

m	$\cos m\pi$	$-2 \cos \frac{m\pi}{2}$	$ $	Total
1	-1	0		0
2	1	2		4
3	-1	0		0
4	1	-2		0
5	-1	0		0
6	1	2		4

So $A_m = \frac{8V_0}{m\pi}$ if $m = 2, 6, 10, 14, \dots$

$$V(x, y) = \frac{8V_0}{\pi} \sum_n \frac{1}{n} e^{-k_n x} \sin k_n y$$

3.19

$$V(R, \theta) = K \cos 3\theta$$

Trig identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Legendre Polynomials

$$P_0 = 1 \quad P_1 = \cos\theta \quad P_2 = \frac{1}{2}(3\cos^2\theta - 1)$$

$$P_3 = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

$$2P_3 = 5\cos^3\theta - 3P_1$$

$$\cos^3\theta = \frac{1}{5}(2P_3 + 3P_1)$$

$$\cos 3\theta = \frac{4}{5}(2P_3 + 3P_1) - 3P_1$$

$$= \frac{8}{5}P_3 - \frac{3}{5}P_1$$

$$V(R, \theta) = \frac{K}{5}(8P_3 - 3P_1)$$

Outside Discard r^n terms

$$V_o(r, \theta) = \sum_n A_n r^{-(n+1)} P_n(\cos \theta)$$

$$\Leftarrow \text{Let } a = R$$

$$V_o(a, \theta) = \frac{8k}{5} P_3 - \frac{3k}{5} P_1 = \sum_n A_n^o a^{-(n+1)} P_n$$

$$\frac{A_1^o}{a^2} = -\frac{3k}{5}$$

$$\frac{A_3^o}{a^4} = \frac{8k}{5}$$

$$V_o(r, \theta) = -\frac{3ka^2}{5r^2} P_1(\cos \theta) + \frac{8ka^4}{5r^4} P_3(\cos \theta)$$

Inside discard $r^{-(n+1)}$ terms

$$V_i(a, \theta) = \sum_n A_n^i r^n P_n(\cos \theta)$$

$$= \frac{8k}{5} P_3 - \frac{3k}{5} P_1$$

$$a^3 A_3^i = \frac{8k}{5}$$

$$a A_1^i = -\frac{3k}{5}$$

$$V_i(r, \theta) = -\frac{3K}{5a} r P_1(\cos\theta) + \frac{8K}{5a^3} r^3 P_3(\cos\theta)$$

Surface Charge Density

$$\frac{\partial V_o}{\partial r} \Big|_a - \frac{\partial V_i}{\partial r} \Big|_a = \frac{-\sigma}{\epsilon_0}$$

$$\frac{\partial V_o}{\partial r} \Big|_a = \frac{6Ka^2}{5a^3} P_1 - \frac{32Ka^4}{5a^5} P_3$$

$$\frac{\partial V_i}{\partial r} \Big|_a = -\frac{3K}{5a} P_1(\cos\theta) + \frac{24Ka^2}{5a^3} P_3(\cos\theta)$$

$$\frac{\partial V_o}{\partial r} \Big|_a - \frac{\partial V_i}{\partial r} \Big|_a = \frac{K}{a} \left(\frac{9}{5} P_1 - \frac{56}{5} P_3 \right) = \frac{-\sigma}{\epsilon_0}$$

$$\sigma = \frac{K\epsilon_0}{5a} (56 P_3 - 9 P_1)$$

3.23

Sphere with uniform surface
charge density

$$\sigma(\theta) = \begin{cases} \sigma_0 & 0 \leq \theta \leq \pi/2 \\ -\sigma_0 & \pi/2 < \theta < \pi \end{cases}$$

Outside (Discard r^n)

$$V_o(r, \theta) = \sum A_n^o r^{-(n+1)} P_n(\cos \theta)$$

$$\left. \frac{\partial V_o}{\partial r} \right|_a = \sum -(n+1) A_n^o a^{-(n+2)} P_n(\cos \theta)$$

Inside (Discard $r^{-(n+1)}$)

$$V_i(r, \theta) = \sum A_n^i r^n P_n(\cos \theta)$$

$$\left. \frac{\partial V_i}{\partial r} \right|_a = \sum n A_n^i a^{n-1} P_n(\cos \theta)$$

Potential Continuous at a

$$V_i(a, \theta) = V_o(a, \theta)$$

$$\sum_n A_n^i a^n P_n(\cos \theta) = \sum_n A_n^o a^{-(n+1)} P_n(\cos \theta)$$

By orthogonality, the series must be equal term by term.

$$A_n^i a^n = A_n^o a^{-(n+1)}$$

$$A_n^o = a^{2n+1} A_n^i$$

Electrostatic Boundary Conditions at a

$$\left. \frac{\partial V_o}{\partial r} \right|_a - \left. \frac{\partial V_i}{\partial r} \right|_a = \frac{-\sigma}{\epsilon_0}$$

$$\begin{aligned} \sum_n -(n+1) A_n^o a^{-(n+2)} P_n(\cos \theta) \\ - \sum_n n A_n^i a^{n-1} P_n(\cos \theta) = \frac{-\sigma(x)}{\epsilon_0} \end{aligned}$$

$$-\sigma(x) = \sum_n P_n(\cos\theta) \left[-A_n^i a^{-(n+2)} (n+1) - n A_n^i a^{(n-1)} \right] \epsilon_0$$

$$= \sum_n A_n^i \epsilon_0 P_n(\cos\theta) \left[-(n+1) a^{-(n+2)} a^{2n+1} - n a^{(n-1)} \right]$$

$$= \sum_n \epsilon_0 A_n^i P_n(\cos\theta) a^{(n-1)} [-2n-1]$$

$$\sigma(x) = \epsilon_0 \sum_n (2n+1) A_n^i P_n(\cos\theta) a^{(n-1)}$$

Multiply by P_m and integrate

$$\int_{-1}^1 \sigma(\theta) P_m(\cos\theta) d(\cos\theta) = \epsilon_0 \sum_n (2n+1) a^{n-1} A_n^i \times \underbrace{\int_{-1}^1 P_m(\cos\theta) P_n(\cos\theta) d(\cos\theta)}_{\text{orthogonality condition}}$$

$$\frac{2}{2m+1} \delta_{nm}$$

$$\int_{-1}^1 \sigma(\theta) P_m(\cos \theta) d(\cos \theta) = \epsilon_0 \sum_n (2n+1) A_n^i a^{n-1} \cdot \frac{2}{2n+1} \delta_{nm}$$

$$= 2 \epsilon_0 A_m^i a^{m-1}$$

Work on the integral

$$\int_{-1}^1 \sigma(\theta) P_m(\cos \theta) d(\cos \theta) = - \int_{\pi}^0 \sigma(\theta) P_m(\cos \theta) \sin \theta d\theta$$

$$= \int_0^{\pi} \sigma(\theta) \sin \theta P_m(\cos \theta) d\theta \equiv I_m$$

Boundary Condition

$$\sigma(\theta) = \begin{cases} \sigma_0 & 0 \leq \theta \leq \pi/2 \\ -\sigma_0 & \pi/2 \leq \theta \leq \pi \end{cases}$$

$$I_m = \sigma_0 \int_0^{\pi/2} P_m(\cos \theta) \sin \theta d\theta - \sigma_0 \int_{\pi/2}^{\pi} P_m(\cos \theta) d(\cos \theta)$$

$$u = \cos \theta \quad du = -\sin \theta d\theta$$

$$I_m = -\sigma_0 \int_{-1}^0 P_m(u) du + \sigma_0 \int_0^1 P_m(u) du$$

$$= \sigma_0 \int_0^1 P_m(u) du - \sigma_0 \int_{-1}^0 P_m(u) du$$

Evidently $I_m = 0$ for even functions, and

$$I_m = 2\sigma_0 \int_0^1 P_m(u) du$$

for odd functions.

~~$I_0 = 2\sigma_0 \int_0^1 P_0 du = 2\sigma_0 \int_0^1 du = 2\sigma_0$~~

$$P_0 = 1, \quad I_0 = 0 \quad \text{since even.}$$

$$P_1 = x, \quad I_1 = 2\sigma_0 \int_0^1 x dx = \sigma_0$$

~~$P_2 = \frac{1}{2}(3x^2 - 1)$~~ , $I_2 = 0$ since even.

$$P_3 = \frac{1}{2}(5x^3 - 3x) \quad I_3 = \sigma_0 \int_0^1 (5x^3 - 3x) dx$$

$$= \sigma_0 \left(\frac{5}{4} - \frac{3}{2} \right) = -\frac{\sigma_0}{4}$$

$$I_0 = 0 \Rightarrow \boxed{A_0 = 0}$$

$$I_1 = \sigma_0 = 2\epsilon_0 A_1^i a^{l-1}$$

$$\boxed{A_1^i = \frac{\sigma_0}{2\epsilon_0}}$$

$$I_2 = 0 \Rightarrow \boxed{A_2 = 0}$$

$$I_3 = -\frac{\sigma_0}{4} = 2\epsilon_0 A_3^i a^{3-1}$$

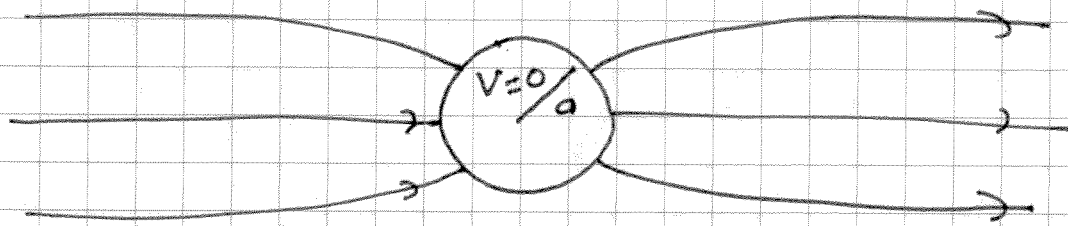
$$\boxed{A_3^i = -\frac{\sigma_0}{8\epsilon_0 a^2}}$$

Outside

$$A_0^o = 0 \quad A_1^o = a^3 A_1^i = \frac{\sigma_0 a^3}{2\epsilon_0}$$

$$A_2^o = 0 \quad A_3^o = a^7 A_3^i = -\frac{\sigma_0 a^5}{8\epsilon_0}$$

E.4.1



$$\vec{E} = E_0 \hat{x} \quad \text{as} \quad x \rightarrow \pm \infty$$

$$V = -E_0 x = -E_0 s \cos \phi$$

The general solution is

$$V(s, \phi) = \sum_n A_n s^{-n} \cos n\phi + B_n s^{-n} \sin n\phi \\ + C_n s^n \cos n\phi + D_n s^n \sin n\phi$$

As $s \rightarrow \infty$, $V \rightarrow -E_0 s \cos \phi$ so

$$C_1 = -E_0, \quad C_0, C_{n \geq 2} = 0 \quad \text{and} \quad D_n = 0$$

$$V(s, \phi) = \sum_n A_n s^{-n} \cos n\phi + B_n s^{-n} \sin n\phi \\ - E_0 s \cos \phi$$

Apply Boundary Condition

$$V(a, \phi) = 0 = \sum A_n a^{-n} \cos n\phi + B_n a^{-n} \sin n\phi - E_0 a \cos \phi$$

By orthogonality, $A_n = 0$ $n \neq 1$, $B_n = 0$

$$A_1 a^{-1} = +E_0 a$$

$$A_1 = +a^2 E_0$$

$$V(s, \phi) = -E_0 s \cos \phi + \frac{E_0 a^2}{s} \cos \phi$$

Electric Field

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial s} \hat{s} - \frac{1}{s} \frac{\partial V}{\partial \phi} \hat{\phi}$$

$$= \left(E_0 \cos \phi + \frac{E_0 a^2}{s^2} \cos \phi \right) \hat{s}$$

$$\frac{1}{s} \left(-E_0 s + \frac{E_0 a^2}{s} \right) \sin \phi \hat{\phi}$$

$$\text{At } s=0, \quad \vec{E} = 2E_0 \cos \phi \hat{s}$$

Charge Density (Gaussian Pillbox)

$$\phi = |E|A = \frac{Q_{enc}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$\sigma = \epsilon_0 |E| = 2 \epsilon_0 E_0 \cos \phi$$

E. 4.2

Outside

$$V_0(s, \phi) = \sum_n A_n s^{-n} \cos n\phi + B_n s^{-n} \sin n\phi$$

By orthogonality, only $\cos 3\phi$ survives.

$$V_0(s, \phi) = \frac{A_3}{s^3} \cos 3\phi$$

B.C

$$V(a, \phi) = V_0 \cos 3\phi = \frac{A_3}{a^3} \cos 3\phi$$

$$A_3 = V_0 a^3$$

$$V_0(s, \phi) = \frac{V_0 a^3}{s^3} \cos 3\phi$$

● Inside

$$V_i(s, \phi) = \sum_n C_n s^n \cos n\phi + D_n s^n \sin n\phi$$

by orthogonality

$$V_i(s, \phi) = C_3 s^3 \cos 3\phi$$

Boundary Conditions

●
$$V(a, \phi) = V_0 \cos 3\phi = C_3 a^3 \cos 3\phi$$

$$C_3 = V_0 / a^3$$

$$V_i(s, \phi) = \frac{V_0 s^3}{a^3} \cos 3\phi$$

Field

$$\nabla f = \hat{s} \frac{\partial f}{\partial s} + \hat{\phi} \frac{1}{s} \frac{\partial f}{\partial \phi}$$

Field Inside

$$\vec{E}_i = -\nabla V_i = -\left(\frac{3V_0 s^2}{a^3} \cos 3\phi \hat{s} - \frac{3V_0 s^2}{a^3} \sin 3\phi \hat{\phi} \right)$$

Field Outside

$$\vec{E}_o = -\nabla V_o = -\left(-\frac{3V_0 a^3}{s^4} \cos 3\phi \hat{s} - \frac{3V_0 a^3}{s^4} \sin 3\phi \hat{\phi} \right)$$

or re-writing

$$\vec{E}_i = \frac{3V_0 s^2}{a^3} \left(-\cos 3\phi \hat{s} + \sin 3\phi \hat{\phi} \right)$$

$$\vec{E}_o = \frac{3V_0 a^3}{s^4} \left(\cos 3\phi \hat{s} + \sin 3\phi \hat{\phi} \right)$$

● Surface Charge Use Gaussian pillbox at surface

$$\sigma = \epsilon_0 \hat{n} \cdot (\vec{E}_o - \vec{E}_i) \quad \hat{n} = \hat{S}$$

$$= \epsilon_0 \left(\frac{3V_0 a^3}{a^4} \cos 3\phi - \frac{3V_0 \sigma^2}{a^3} (-\cos 3\phi) \right)$$

$$= \frac{6V_0 \epsilon_0}{a} \cos 3\phi$$