Homework 6

Due Friday 3/8/2013 - at beginning of class

Griffiths' 4 Problems (3rd Edition numbers are the same)

4.5

4.10

4.15

4.18

4.20

4.21

Additional Problems

E.6.1 A dipole formed of a $+Q$ and $-Q$ charge spaced a distance a apart has dipole moment pointing in the $+\hat{z}$ direction. The center of the dipole is located at $+R\hat{z}$ a distance R above a neutral dielectric slab occupying the volume $z < 0$ with dielectric constant κ. Compute the force the dielectric plane exerts on the dipole.

E.6.2 A point charge $+Q$ is a distance R above a neutral dielectric slab with dielectric constant κ occupying the volume $z < 0$. Compute the electric field immediately above and below the dielectric surface. From the field, calculate the bound charge density at the surface.

E.6.3 A linear dielectric with dielectric constant κ occupies the volume $-a < z < a$. A uniform volume charge density ρ is fixed within the dielectric. Compute the electric field everywhere. Compute the polarization everywhere.

4.5 Torque on dipole $\vec{\tau} = \vec{p} \times \vec{E}$

Electric field of dipole, $\vec{p} = p\hat{y}$, along axis.

<u>x-axis</u> $\vec{E} = -\frac{kp}{x^3}\hat{y}$ <u>y-axis</u> $\vec{E} = \frac{2kp}{y^3}\hat{y}$

<u>Torque on p_1 due to p_2</u>

$$\vec{E}_{21} = \frac{2kp_2}{r^3}\hat{x} \qquad \vec{p}_1 = p_1\hat{y}$$

$$\vec{\tau}_{21} = \vec{p}_1 \times \vec{E}_{21} = (p_1\hat{y}) \times \left(\frac{2kp_2}{r^3}\hat{x}\right)$$

$$= -\frac{2kp_1p_2}{r^3}\hat{z}$$

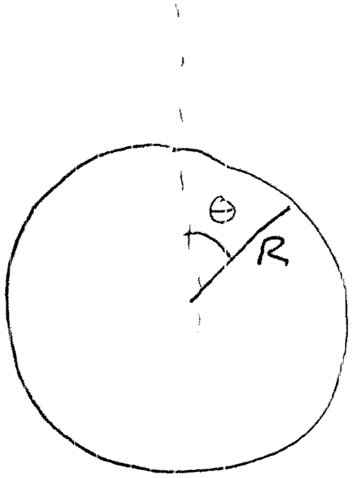
<u>Torque on p_2 due to p_1</u>

$$\vec{E}_{12} = -\frac{kp_1}{r^3}\hat{y} \qquad \vec{p}_2 = p_2\hat{x}$$

$$\vec{\tau}_{12} = \vec{p}_2 \times \vec{E}_{12} = (p_2\hat{x}) \times \left(-\frac{kp_1}{r^3}\hat{y}\right)$$

$$= -\frac{kp_1p_2}{r^3}\hat{z}$$

4.10



(a)

$$\rho_b = -\nabla \cdot \vec{P} = -\nabla \cdot k(x, y, z)$$

$$= -3k$$

$$\sigma_b = \vec{P} \cdot \hat{n} = \vec{P} \cdot \hat{r} = (kR\hat{r}) \cdot \hat{r} = kR$$

(b) We have a system with a uniform surface charge $\sigma = kR$ and a uniform volume charge $\rho = -3k$. Apply Gauss Law.

<u>Inside Sphere</u> $r < R$

$$Q_{enc} = \frac{4}{3}\pi r^3 \rho = -\frac{4}{3}\pi 3k r^3 = -4\pi k r^3$$

<u>Gauss</u> $\Phi = 4\pi r^2 E = \dfrac{Q_{enc}}{\varepsilon_0}$

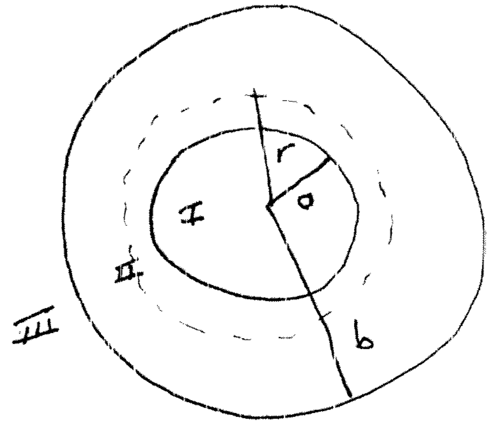
$$\vec{E} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{-4/3\,\pi 3k r^3}{4\pi\varepsilon_0 r^2}\hat{r} = -\frac{kr}{\varepsilon_0}\hat{r}$$

Outside Sphere $\quad Q_{enc} = \left(\frac{4}{3}\pi R^3\right)\underbrace{(-3k)}_{\rho} + \left(4\pi R^2\right)\underbrace{(kR)}_{\sigma}$

$$Q_{enc} = -4\pi k R^3 + 4\pi k R^3 = 0$$

$$\vec{E} = 0$$

4.15



$$\vec{P} = \frac{k}{r}\hat{r}$$

(a) Bound charge

$$\sigma_b = \vec{P}\cdot\hat{n}$$

<u>Inner Surface</u> $\hat{n} = -\hat{r}$

$$\sigma_b = -\frac{k}{a} \qquad Q_{inner} = 4\pi a^2 \sigma_b = -4\pi k a$$

<u>Outer Surface</u> $\hat{n} = \hat{r}$

$$\sigma_b = \frac{k}{b} \qquad Q_{outer} = 4\pi b^2 \sigma_b = 4\pi k b$$

<u>Volume Charge</u>

$$\rho_b = -\nabla\cdot\vec{P} = -k\,\nabla\cdot\left(\frac{\hat{r}}{r}\right)$$

$$= \frac{-k}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{1}{r}\right) = -\frac{k}{r^2}$$

Region III $\quad Q_{enc} = Q_{inner} + Q_{vol}(b) + Q_{outer}$

$$= -4\pi k a - 4\pi k (b-a) + 4\pi k b$$

$$= 0$$

$$\vec{E}_{III} = 0$$

(b) There is no free charge,

$$\nabla \cdot \vec{D} = 0 \quad \Rightarrow \quad \vec{D} = 0$$

(constant solution violates spherical symmetry)

In region I, III $\quad \vec{P} = 0$

$$\vec{D} = 0 = \varepsilon_0 \vec{E} + \vec{P} \Rightarrow \vec{E} = 0$$

$$\vec{E}_I = 0 \qquad \vec{E}_{III} = 0$$

In region II,

$$\vec{D} = 0 = \varepsilon_0 \vec{E}_{II} + \vec{P}$$

$$\vec{E}_{II} = -\frac{\vec{P}}{\varepsilon_0} = -\frac{k}{\varepsilon_0 r}\hat{r}$$

Gauss Law

Region I $r < a$ $Q_{enc} = 0$ $\vec{E}_I = 0$

Region II $a < r < b$ $Q_{enc} = Q_{inner} + Q_{vol}(r)$

$$Q_{vol}(r) = \int_a^r 4\pi r^2 \rho_b \, dr$$

$$= \int_a^r 4\pi r^2 \left(-\frac{k}{r^2} \right) dr$$

$$= -4\pi k (r - a)$$

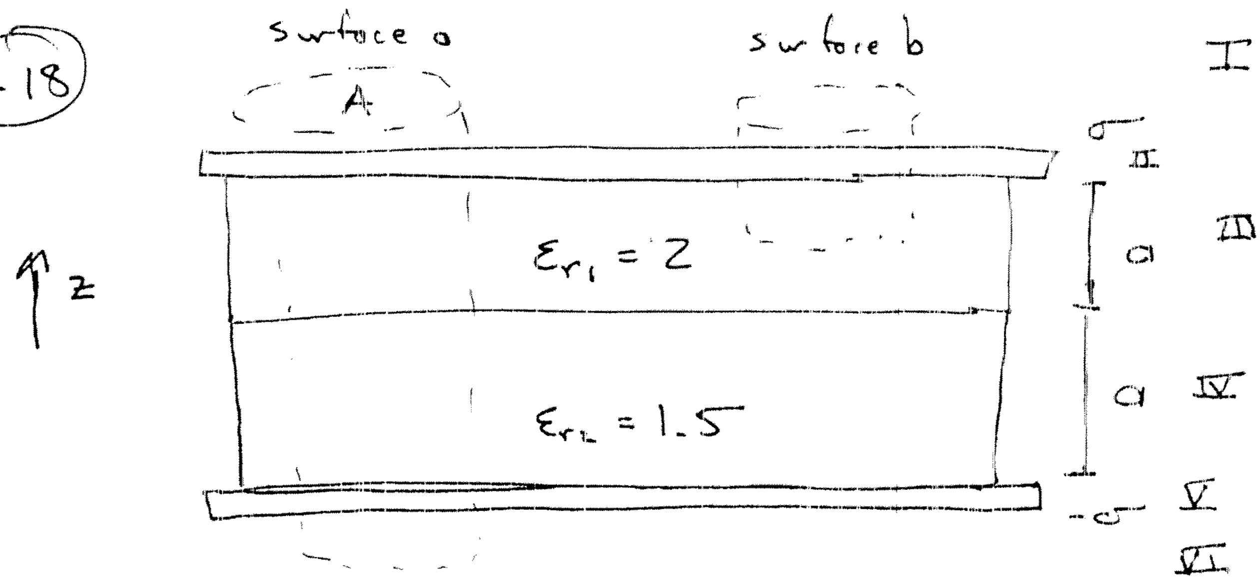
$$Q_{enc} = -4\pi k a - 4\pi k (r-a) = -4\pi k r$$

Gauss $\Phi = 4\pi r^2 E = Q_{enc}/\varepsilon_0$

$$\vec{E}_{II} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} \hat{r} = \frac{-4\pi k r}{4\pi\varepsilon_0 r^2} \hat{r}$$

$$\vec{E}_{II} = \frac{-k}{\varepsilon_0 r} \hat{r}$$

(4.18)



(a) The total free charge in surface a is

$$Q_{fenc} = \sigma A + (-\sigma A) = 0$$

by symmetry $\vec{D}_{I} = \vec{D}_{VI} = 0$

The free charge in surface b is $Q_{fenc} = \sigma A$

Apply Gauss Law

$$\Phi = D_{I} A - D_{III} A = Q_{fenc}$$

$$\vec{D}_{III} = -\sigma \hat{z} = \vec{D}_{IV}$$

Naturally, $\vec{D}_{II}$ and $\vec{D}_{V} = 0$

(b) $\vec{D}_{III} = \varepsilon_{r1}\varepsilon_0 \vec{E}_{III}$

$$\vec{E}_{III} = \frac{-\sigma}{\varepsilon_{r1}\varepsilon_0 \cancel{\varepsilon_0}}\hat{z} = \frac{-\sigma}{2\varepsilon_0}\hat{z} \quad \text{top slab}$$

$$\vec{E}_{IV} = \frac{-\sigma}{\varepsilon_{r2}\varepsilon_0}\hat{z} = -\frac{2}{3}\frac{\sigma}{\varepsilon_0}\hat{z} \quad \text{bottom slab}$$

(c) <u>Polarization</u>

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

<u>Top Slab</u> $\chi_{e1} = \varepsilon_{r1} - 1 = 1$

$$\vec{P}_{III} = \varepsilon_0 \chi_{e1} \vec{E}_{III} = -\frac{\chi_{e1}\sigma\varepsilon_0}{\varepsilon_{r1}\varepsilon_0}\hat{z}$$

$$= -\frac{\chi_{e1}}{\varepsilon_{r1}}\sigma\hat{z} = -\frac{\chi_{e1}}{(\chi_{e1}+1)}\sigma\hat{z}$$

$$= -\frac{1}{2}\sigma\hat{z}$$

$\chi_{e2} = 3/2 - 1 = 1/2$

<u>Bottom Slab</u>

$$\vec{P}_{IV} = \frac{-\chi_{e2}}{\varepsilon_{r2}}\sigma\hat{z} = \frac{-1/2}{3/2}\sigma\hat{z}$$

$$= \frac{-1}{3}\sigma\hat{z}$$

(d) $|\Delta V| = E_{III}\, a + E_{IV}\, a$

$$= \frac{\sigma}{2\varepsilon_0} a + \frac{2\sigma}{3\varepsilon_0} a$$

$$= \frac{7}{6} \frac{\sigma}{\varepsilon_0} a$$

(e) Bound charge –

<u>Slab 1</u> top – $\hat{n} = \hat{z}$

$$\sigma_{1,top} = \vec{P} \cdot \hat{z} = -\frac{1}{2}\sigma$$

bottom – $\hat{n} = -\hat{z}$

$$\sigma_{1,bottom} = \frac{1}{2}\sigma$$

<u>Slab 2</u> top $\hat{n} = \hat{z}$

$$\sigma_{2,top} = \vec{P} \cdot \hat{z} = -\frac{1}{3}\sigma$$

$$\sigma_{2,bottom} = \vec{P} \cdot (-\hat{z}) = \frac{1}{3}\sigma$$

(f) Field in top slab, superposition of

$\sigma, -\sigma$ and $-1/2\,\sigma, \frac{1}{2}\sigma$

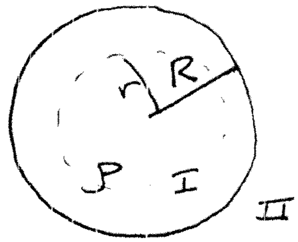
$$\vec{E}_{III} = -\frac{\sigma}{\varepsilon_0}\hat{z} + \frac{\frac{1}{2}\sigma}{\varepsilon_0}\hat{z} = -\frac{1}{2}\frac{\sigma}{\varepsilon_0}\hat{z}$$

Field in bottom slab superposition of

$\sigma, -\sigma$ and $-\frac{1}{3}\sigma, \frac{1}{3}\sigma$

$$\vec{E} = -\frac{\sigma}{\varepsilon_0}\hat{z} + \frac{\frac{1}{3}\sigma}{\varepsilon_0}\hat{z} = -\frac{2}{3}\frac{\sigma}{\varepsilon_0}\hat{z}$$

4.20



Gauss Low

Region I $\quad r < R$

$$Q_{enc} = \frac{4}{3}\pi r^3 \rho$$

$$\Phi = 4\pi r^2 D = Q_{fenc}$$

$$\vec{D}_I = \frac{\frac{4}{3}\pi r^3 \rho}{4\pi r^2}\hat{r} = \frac{\rho r}{3}\hat{r}$$

$$\vec{D}_I = \varepsilon_r \varepsilon_0 \vec{E}_I$$

$$\vec{E}_I = \frac{\rho r}{3\varepsilon_r \varepsilon_0}\hat{r} = \frac{\rho r}{3\varepsilon_r \varepsilon_0}\hat{r}$$

Region II $\quad r > R$

$$Q_{fenc} = \frac{4}{3}\pi R^3 \rho$$

$$\vec{D}_{II} = \frac{Q_{fenc}}{4\pi r^2}\hat{r} = \frac{\frac{4}{3}\pi R^3 \rho}{4\pi r^2}\hat{r}$$

$$= \frac{R^3 \rho}{3 r^2}\hat{r}$$

$$\vec{D}_{II} = \varepsilon_0 \vec{E}_{II}$$

$$\vec{E}_{II} = \frac{R^3 \rho}{3\varepsilon_0 r^2}\hat{r}$$

Potential Difference

$$\Delta V_{\infty 0} = -\int_{\infty}^{0} \vec{E}\cdot d\vec{l} \qquad = -\int_{\infty}^{R} \vec{E}_{II}\cdot d\vec{l} \; - \int_{R}^{0} \vec{E}_{I}\cdot d\vec{l}$$

$$d\vec{l} = dr\,\hat{r} \qquad (dr<0 \text{ because of limits})$$

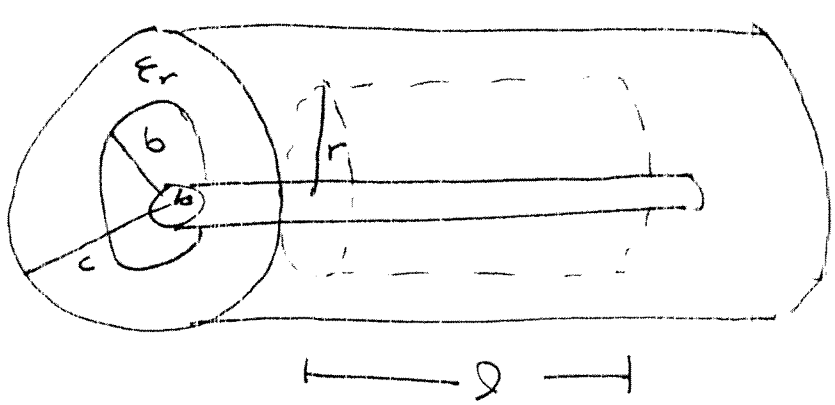
$$\Delta V_{\infty 0} = -\int_{\infty}^{R} \frac{R^3 \rho}{3\varepsilon_0 r^2}\,dr \; - \int_{R}^{0} \frac{\rho r}{3\varepsilon_r \varepsilon_0}\,dr$$

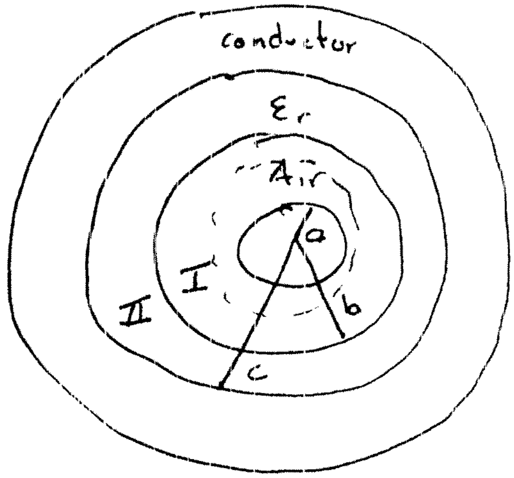
$$= \left.\frac{R^3 \rho}{3\varepsilon_0 r}\right|_{\infty}^{R} - \left.\frac{\rho r^2}{6\varepsilon_r \varepsilon_0}\right|_{R}^{0}$$

$$\Delta V_{\infty 0} = \frac{R^3 \rho}{3 \varepsilon_0} \left(\frac{1}{R} - \frac{1}{\infty} \right) + \frac{\rho R^2}{6 \varepsilon_0 \varepsilon_r}$$

$$= \frac{\rho R^2}{3 \varepsilon_0} \left(1 + \frac{1}{2 \varepsilon_r} \right)$$

4.21





Add +Q charge per length ℓ of the Gaussian surface.

Q_{fenc} in cylindrical Gaussian surface of radius r

is $Q_{fenc} = Q$

Gauss Law $\quad \Phi_d = 2\pi r \ell D = Q_{fenc}$

$$\vec{D} = \frac{Q}{\ell} \cdot \frac{1}{2\pi r}$$

Region I $\quad a < r < b$

$$\vec{D} = \varepsilon_0 \vec{E}_I$$

$$\vec{E}_I = \frac{Q}{\ell} \cdot \frac{1}{2\pi\varepsilon_0 r} \hat{r}$$

Note, using $\hat{r} = \hat{s}$ because of habit. Definitely a cylindrical problem.

Region II $\quad b < r < c$

$$\vec{D} = \varepsilon_r \varepsilon_0 \vec{E}_{II}$$

$$\vec{E}_{II} = \frac{Q}{\ell} \frac{1}{2\pi\varepsilon_0 \varepsilon_r r} \hat{r}$$

Potential Difference

$$|\Delta V| = |\Delta V_I| + |\Delta V_{II}| \quad \text{(fields in same direction)}$$

$$|\Delta V_I| = \left| - \int_a^b \vec{E}_I \cdot d\vec{\ell} \right| \qquad d\vec{\ell} = \hat{r}\, dr$$

$$= \left| - \frac{Q}{\ell} \cdot \frac{1}{2\pi\varepsilon_0} \int_a^b \frac{dr}{r} \right| = \frac{Q}{\ell} \cdot \frac{1}{2\pi\varepsilon_0} \ln\left(b/a\right)$$

$$|\Delta V_{II}| = \left| -\int_b^c \vec{E}_{II} \cdot d\vec{l} \right| = \left| -\frac{Q}{l} \frac{1}{2\pi\varepsilon_0\varepsilon_r} \int_b^c \frac{dr}{r} \right|$$

$$= \frac{Q}{l} \frac{1}{2\pi\varepsilon_0\varepsilon_r} \ln(c/b)$$

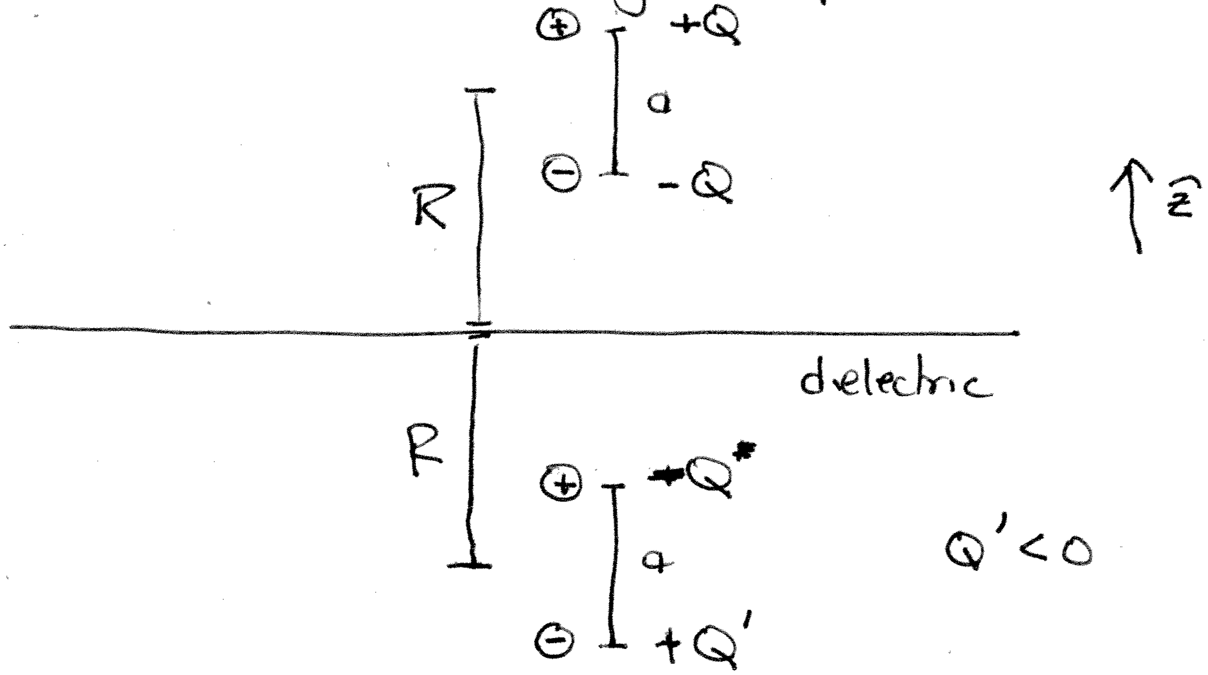
$$|\Delta V| = |\Delta V_I| + |\Delta V_{II}|$$

$$= \frac{Q}{2\pi\varepsilon_0 l} \left(\ln(b/a) + \frac{1}{\varepsilon_r} \ln(c/b) \right)$$

$$C = \frac{Q}{|\Delta V|} = \frac{2\pi\varepsilon_0 l}{\ln(b/a) + \frac{1}{\varepsilon_r} \ln(c/b)}$$

$$\frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln(b/a) + \frac{1}{\varepsilon_r} \ln(c/b)}$$

E.6.1 Use image dipole



$Q' < 0$

$$Q' = -\left(\frac{\kappa - 1}{\kappa + 1}\right) Q = -\frac{\chi_e}{\chi_e + 2} Q \equiv -\gamma Q$$

<u>Force</u>

$$\vec{F}_{dip} = \frac{-kQQ'}{(2R)^2} - \frac{kQQ'}{(2R)^2} + \frac{kQQ'}{(2R+a)^2} + k\frac{QQ'}{(2R-a)^2}$$

$$= k\gamma Q^2 \left[\frac{1}{4R^2} + \frac{1}{4R^2} - \frac{1}{(2R+a)^2} - \frac{1}{(2R-a)^2}\right] \times \hat{z}$$

$$= -k\gamma Q^2 \frac{(12a^2R^2 - a^4)}{2R^2(2R-a)^2(2R+a)^2}$$

(alpha)

WolframAlpha computational knowledge engine

simplify 1/(2*r^2)-1/(2r-a)^2-1/(2r+a)^2

Examples Random

Input interpretation:

simplify $\frac{1}{2r^2} - \frac{1}{(2r-a)^2} - \frac{1}{(2r+a)^2}$

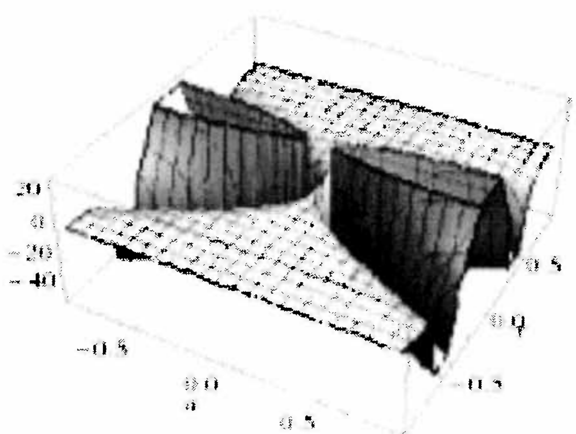
Results: More

$$-\frac{1}{(a+2r)^2} - \frac{1}{(a-2r)^2} + \frac{1}{2r^2}$$

$$\frac{a^4 - 12a^2r^2}{2(a^2r - 4r^3)^2}$$

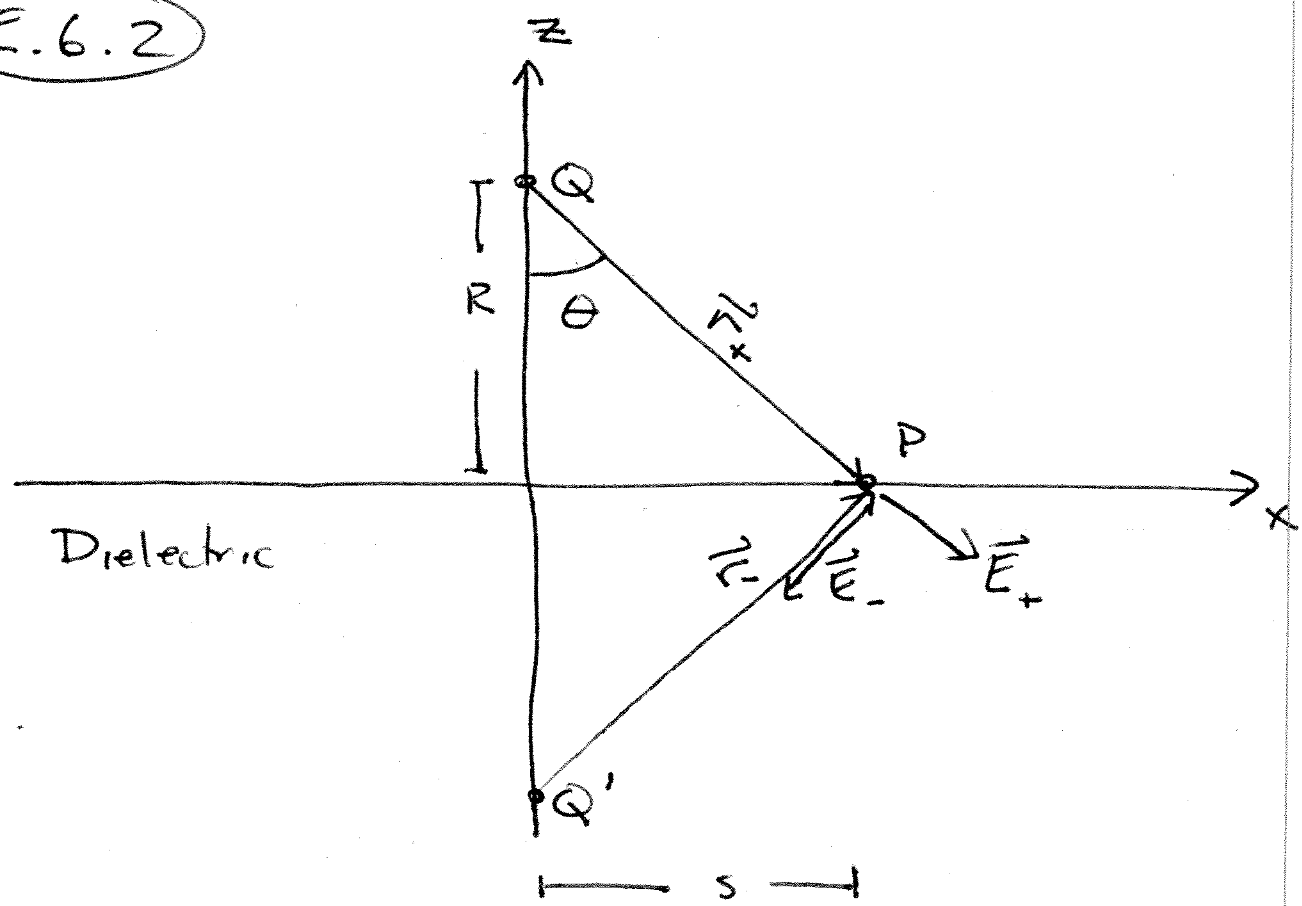
$$-\frac{12a^2r^2 - a^4}{2r^2(a-2r)^2(a+2r)^2}$$

3D plot: Show contour lines



Enable interactivity

E.6.2



For $\vec{E}_t$ the field immediately above the plane, $\vec{E}_t$

$$\vec{E}_t = \vec{E}_+ + \vec{E}_- = \cancel{2|\vec{E}_+|\cos\theta}$$

$$= \frac{kQ}{r_+^3}\vec{r}_+ + \frac{kQ'}{r_-^3}\vec{r}_-$$

$$Q' = -\left(\frac{k-1}{k+1}\right)Q \equiv -\gamma Q$$

$$\gamma = \frac{k-1}{k+1}$$

$$\vec{r}_+ = s\hat{s} - R\hat{z}$$

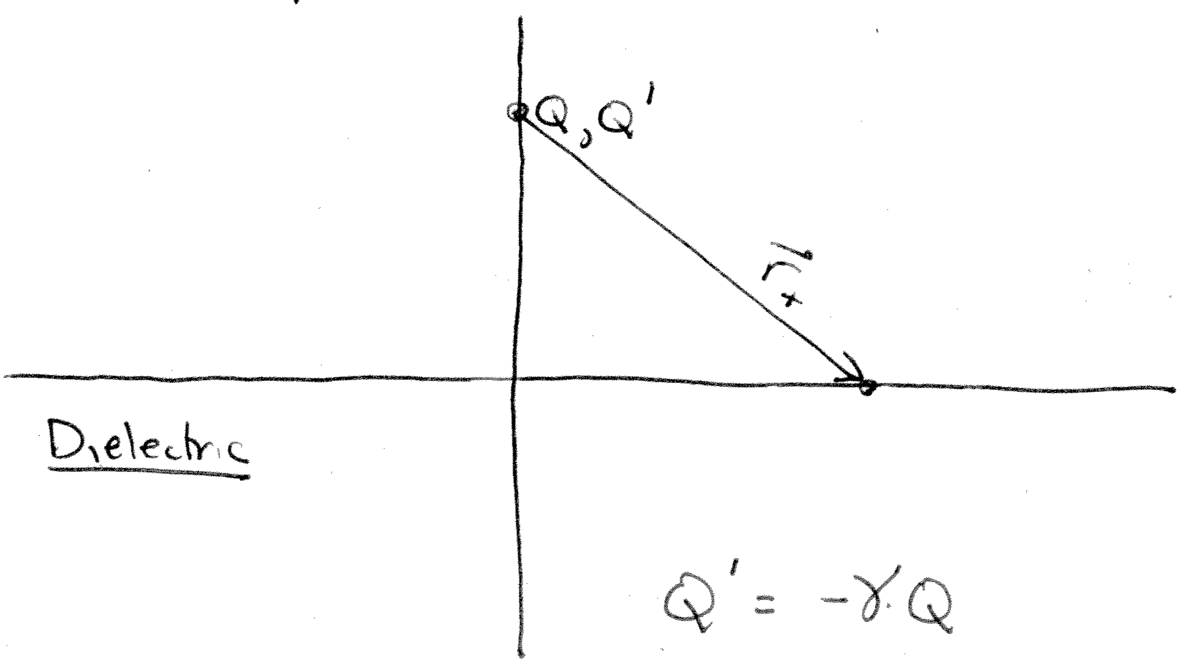
$$r_+ = \sqrt{s^2 + R^2} = r_-$$

$$\vec{r}_- = s\hat{s} + R\hat{z}$$

$$\vec{E}_t = \frac{KQ}{(s^2+R^2)^{3/2}}\left(s\hat{s} - R\hat{z}\right) - \frac{K\gamma Q}{(s^2+R^2)^{3/2}}\left(s\hat{s} + R\hat{z}\right)$$

$$= \frac{KQ}{(s^2+R^2)^{3/2}}\left[s(1-\gamma)\hat{s} - (\gamma+1)R\,\hat{z}\right]$$

Below the plane



$$Q' = -\gamma \cdot Q$$

$$\vec{E}_b = \frac{k(Q+Q')}{r_+^3}\vec{r}_+$$

$$= \frac{kQ(1-\gamma)}{(s^2+R^2)^{3/2}} \cdot (s\hat{s} - R\hat{z})$$

<u>Charge Density</u> Use Gaussian Pillbox at surface.

$$\Phi_e = \vec{E}_t \cdot \hat{z} A - \vec{E}_b \cdot \hat{z} A = \frac{\sigma_b A}{\varepsilon_0}$$

<u>Cancel A</u>

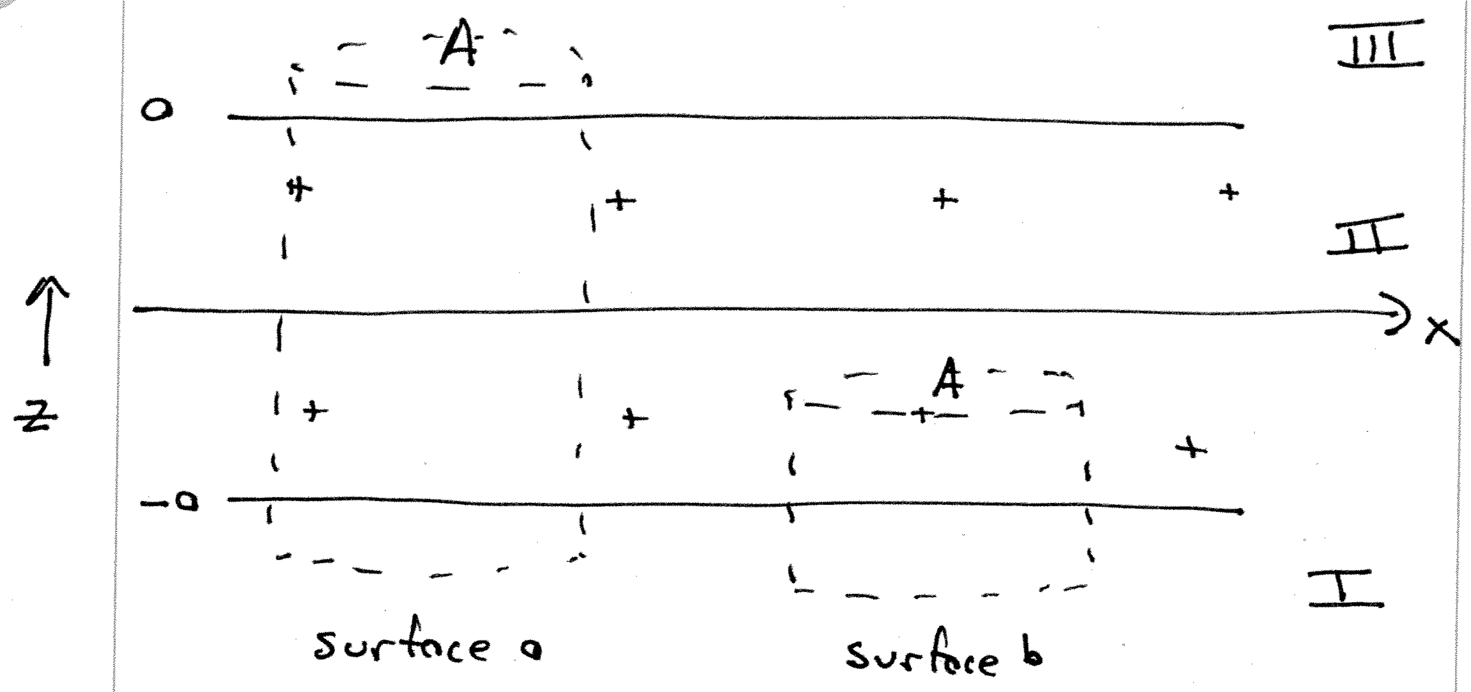
$$\vec{E}_t \cdot \hat{z} = -\frac{kQR(\gamma+1)}{(s^2+R^2)^{3/2}}$$

$$\vec{E}_b \cdot \hat{z} = -\frac{kQR(1-\gamma)}{(s^2+R^2)^{3/2}}$$

$$\sigma_b = \varepsilon_0\left(\frac{-kQR(\gamma+1)}{(s^2+R^2)^{3/2}} + \frac{kQR(1-\gamma)}{(s^2+R^2)^{3/2}}\right)$$

$$= \frac{-QR\gamma}{2\pi(s^2+R^2)^{3/2}} \checkmark$$

E.6.3



Surface a Apply Gauss' Law for Displacement

$$\Phi_d = \vec{D}_{III} \cdot \hat{z} A + \vec{D}_I(-\hat{z})A = Q_{fenc}$$

$$= 2aA\rho$$

$$D_{III} - D_I = 2a\rho$$

By symmetry $D_{III} = -D_I$

$$2D_{III} = 2a\rho$$

$$\vec{D}_{III} = a\rho\hat{z}$$

$$\vec{D}_I = -a\rho\hat{z}$$

Surface b $\quad Q_{fenc} = (x+a)A\rho$

$$\cancel{D} \quad D_{II} - D_{I} = (x+a)\rho = \frac{Q_{fenc}}{A}$$

$$D_{II} = D_{I} + (x+a)\rho$$

$$= -a\rho + (x+a)\rho = x\rho$$

$$\vec{D}_{II} = x\rho\hat{z}$$

Electric Field and Polarization

In region I and III, $\vec{P} = 0$, so

$$\vec{D}_{III} = \varepsilon_0 \vec{E}_{III} \quad \text{and} \quad \vec{D}_{I} = \varepsilon_0 \vec{E}_{I}$$

$$\vec{E}_{I} = \frac{-a\rho}{\varepsilon_0}\hat{z} \qquad \vec{P}_{I} = 0$$

$$\vec{E}_{III} = \frac{a\rho}{\varepsilon_0}\hat{z} \qquad \vec{P}_{III} = 0$$

In region II, $\quad \vec{D}_{II} = \varepsilon_0 \kappa \vec{E}_{II}$ in linear dielectric

$$\vec{E}_{II} = \frac{x\rho}{\kappa\varepsilon_0}\hat{z} \qquad \vec{P}_{II} = \varepsilon_0 \chi_e \vec{E}_{II}$$

$$= \varepsilon_0(\kappa-1)\vec{E}_{II}$$

$$\vec{P}_{II} = \frac{x\rho}{\varepsilon_0}\frac{\kappa-1}{\kappa}\hat{z}$$