Homework 7

Due Thursday 3/14/2013 - at beginning of class

Griffiths' 4 Problems

4.24

4.26

4.39 (Griffiths 3rd edition problem 4.36)

5.4

5.5

5.9

5.11 (Griffiths 3rd edition problem 5.10)

4.24 Let the surface of the sphere be at zero potential.

Solutions to Laplace's Eqn

Conductor
$V_c = 0 \qquad r < a$

Dielectric
$$V_K = \sum A_n r^n P_n(\cos\theta) + B_n r^{-(n+1)} P_n(\cos\theta)$$

cannot throw away terms because r does not go to zero ~~in the~~ or ∞ in the dielectric. $(a < r < b)$

Outside
$$V_o = \sum C_n r^{-(n+1)} P_n(\cos\theta) - E_0 r P_1(\cos\theta)$$

discard r^n terms because they blow up except $V = -E_0 r \cos\theta$
$\Rightarrow \vec{E} = E_0 \hat{z}$

Boundary Conditions

$$V_c(a,\theta) = 0 = V_K(a,0)$$

V_K must be zero term by term

$$A_n a^n + B_n a^{-(n+1)} = 0$$

$$B_n = -A_n a^{2n+1}$$

$$V_K = \sum A_n P_n(\cos\theta)\left(r^n - \frac{a^{2n+1}}{r^{n+1}}\right)$$

Continuous at $r = b$

$$V_K(b,\theta) = V_0(b,\theta)$$

$$\sum A_n P_n(\cos\theta)\left(b^n - \frac{a^{2n+1}}{b^{n+1}}\right)$$

$$= \sum C_n \frac{1}{b^{n+1}} P_n(\cos\theta) - E_0 b P_1$$

$n=1$ $$A_1\left(b - \frac{a^3}{b^2}\right) = \frac{C_1}{b^2} - E_0 b$$

$n \neq 1$ $$A_n\left(b^n - \frac{a^{2n+1}}{b^{n+1}}\right) = \frac{C_n}{b^{n+1}}$$

Electrostatic B.C. at $r=b$

$$\left.\frac{\partial V_0}{\partial r}\right|_b - \varepsilon_r \left.\frac{\partial V_k}{\partial r}\right|_b = -\sigma_f = 0$$

$$\sum -(n+1) C_n \frac{1}{b^{n+2}} P_n(\cos\theta) - E_0 P_1(\cos\theta)$$

$$- \varepsilon_r \sum A_n P_n(\cos\theta)\left(n b^{n-1} + (n+1)\frac{a^{2n+1}}{b^{n+2}}\right)$$

$$= 0$$

$n=1$

$$-2\frac{C_1}{b^3} - E_0 - \varepsilon_r A_1\left(1 + 2\frac{a^3}{b^3}\right) = 0$$

$n \neq 1$

$$-(n+1)\frac{C_n}{b^{n+2}} - \varepsilon_r A_n\left(n b^{n-1} + (n+1)\frac{a^{2n+1}}{b^{n+2}}\right)$$

$$= 0$$

For $n \neq 1$, we have two linear equations in two unknowns,

$$c_1 A_n + c_2 C_n = 0$$

$$c_3 A_1 + c_4 C_n = 0$$

the solution is $A_n = B_n = C_n = 0$

Solve $n = 1$

$$-2C_1 - b^3 E_0 - \varepsilon_r A_1 (b^3 + 2a^3) = 0$$

$$A_1 (b^3 - a^3) = C_1 - E_0 b^3$$

$$C_1 = A_1 (b^3 - a^3) + E_0 b^3$$

$$-2 (A_1 (b^3 - a^3) + E_0 b^3) - b^3 E_0 - \varepsilon_r A_1 (b^3 + 2a^3) = 0$$

$$-A_1 \left(+2 (b^3 - a^3) + \varepsilon_r (b^3 + 2a^3) \right) = 3 b^3 E_0$$

$$A_1 = \frac{-3 b^3 E_0}{2 (b^3 - a^3) + \varepsilon_r (b^3 + 2a^3)}$$

$$A_1 = \frac{-3E_0}{2\left(1 - a^3/b^3\right) + \varepsilon_r\left(1 + \frac{2a^3}{b^3}\right)}$$

<u>Potential in Dielectric</u>

$$V_R = P_1(\cos\theta)\, A_1 \left(r^1 - \frac{a^3}{r^2}\right)$$

$$= \frac{-3E_0 \cos\theta}{2\left(1 - a^3/b^3\right) + \varepsilon_r\left(1 + \frac{2a^3}{b^3}\right)} \left(r - \frac{a^3}{r^2}\right)$$

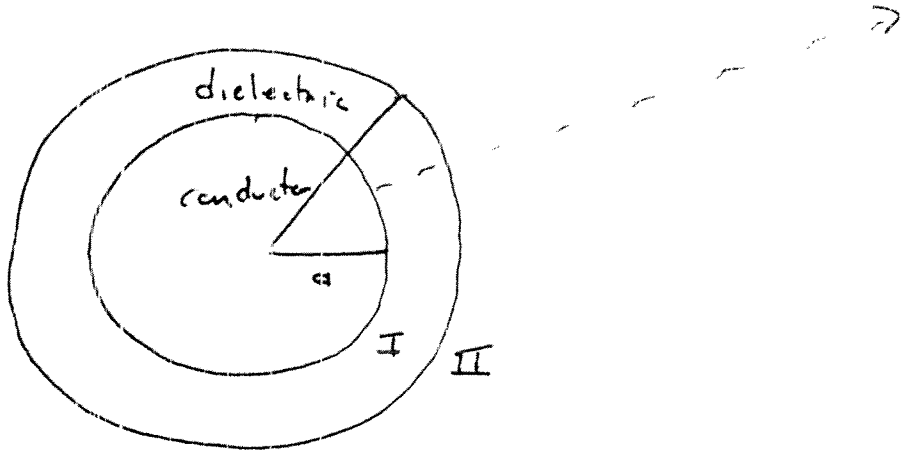
<u>Field in Dielectric.</u>

$$\vec{E} = -\nabla V =$$

$$\frac{+3E_0}{2\left(1 - \frac{a^3}{b^3}\right) + \varepsilon_r\left(1 + \frac{2a^3}{b^3}\right)} \left[\left(1 + \frac{2a^3}{r^3}\right)\cos\theta\, \hat{r} - \left(1 - \frac{a^3}{r^3}\right)\sin\theta\, \hat{\theta}\right]$$

using gradient from front cover.

4.26



Electric field

$$\vec{E}_I = \frac{Q}{4\pi\varepsilon_0\varepsilon_r r^2}\hat{r} \qquad \vec{E}_{II} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

Potential Difference $\Delta V_{a\infty}$

$$\Delta V = -\int_a^b \vec{E}_I \cdot d\vec{l} - \int_b^\infty \vec{E}_{II} \cdot d\vec{l}$$

$$= -\int_a^b \frac{Q}{4\pi\varepsilon_0\varepsilon_r r^2}\,dr - \int_b^\infty \frac{Q}{4\pi\varepsilon_0 r^2}\,dr$$

$$= \frac{Q}{4\pi\varepsilon_0\varepsilon_r}\left(\frac{1}{b} - \frac{1}{a}\right) - \frac{Q}{4\pi\varepsilon_0 b}$$

Capacitance

$$C = \frac{Q}{|\Delta V|} = \frac{4\pi\varepsilon_0}{\frac{1}{\varepsilon_r}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{1}{b}}$$

$$= \frac{4\pi\varepsilon_0\varepsilon_r}{\frac{1}{a} - \frac{1}{b} + \frac{\varepsilon_r}{b}}$$

$$= \frac{4\pi\varepsilon_0\varepsilon_r}{\frac{b - a + \varepsilon_r a}{ab}} = \frac{4\pi\varepsilon_0\varepsilon_r ab}{b + (\varepsilon_r - 1)a}$$

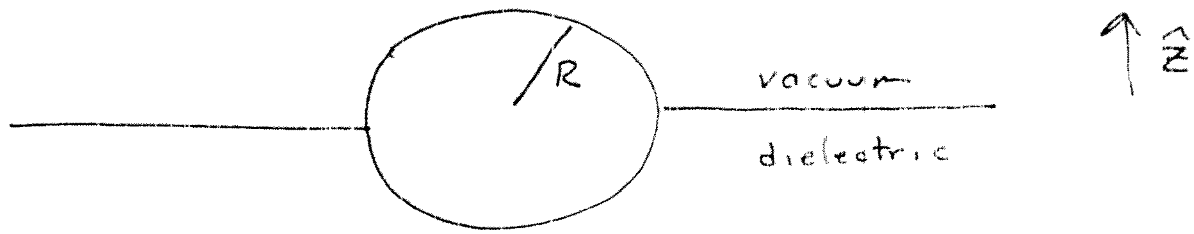
$$= \frac{4\pi\varepsilon_0\varepsilon_r ab}{b - \chi_e a}$$

Energy

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}Q^2\left(\frac{b - \chi_e a}{4\pi\varepsilon_0\varepsilon_r ab}\right)$$

$$= \frac{Q^2}{8\pi\varepsilon_0\varepsilon_r}\left(\frac{1}{a} - \frac{\chi_e}{b}\right)$$

4.34



(a) $V = \dfrac{V_0 R}{r}$ if potential same as missing dielectric.

Field everywhere

$$\vec{E} = -\nabla V = \frac{V_0 R}{r^2}\hat{r}$$

Polarization $\vec{P} = 0, \quad z > 0$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \frac{\varepsilon_0 V_0 R \chi_e}{r^2}\hat{r} \qquad z < 0$$

Bound Charge

$$\rho_b = -\nabla \cdot \vec{P} = 0$$

$$\sigma_b = \vec{P}\cdot\hat{z} = 0 \quad \text{interface between planes}$$

$$\sigma_b = -\vec{P}\cdot\hat{n}\Big|_R = -\frac{\varepsilon_0 V_0 \chi_e}{R} \quad \text{at surface of sphere.}$$

$-$ for inward normal

Free Charge

$z > 0$ Using Gaussian Pillbox

$$\Phi = \vec{E}_c(R)\cdot\hat{n}\,A - 0 = \frac{\sigma_{+c}\,A}{\varepsilon_0}$$

$$\sigma_{c+} = \varepsilon_0\,\vec{E}_0(R) = \frac{\varepsilon_0 V_0}{R}$$

$z < 0$ Pillbox encloses σ_{c-} and σ_b

$$\Phi = \vec{E}_k(R)\cdot\hat{r}A - 0 = \frac{(\sigma_{c-}+\sigma_b)A}{\varepsilon_0}$$

$$\varepsilon_0\left(\frac{V_0 R}{R^2}\right) = \sigma_{c-}+\sigma_b$$

$$\sigma_{c-} = \frac{\varepsilon_0 V_0}{R} + \frac{\varepsilon_0 V_0 \chi_e}{R}$$

$$= \frac{\varepsilon_0 V_0}{R}\,\varepsilon_r$$

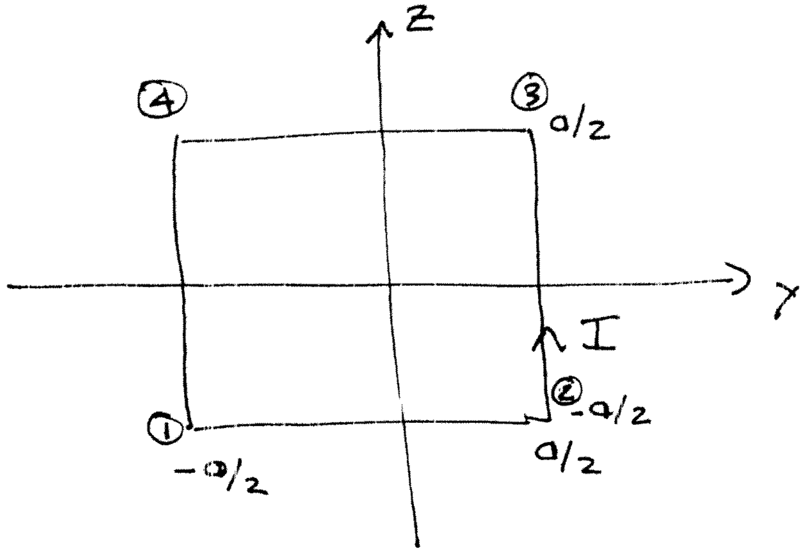
(b) The total charge is σ_{c+} over the entire sphere because $\sigma_{c+} = \sigma_{c-}+\sigma_b$, which does produce the field.

(c) Since we meet the boundary conditions, we must have the solution.

(d) (b) yes , (a) no. We would not meet the electrostatic boundary condition at the boundary.

5.4

$$\vec{B} = kz\,\hat{x}$$



$$\vec{F}_{12} = I\vec{L}\times\vec{B} = I(a\hat{y})\times\left(-\frac{a}{2}k\hat{x}\right)$$

$$= \frac{Ia^2}{2}k\,\hat{z}$$

$$\vec{F}_{23} = \int_{-a/2}^{a/2} I(a\hat{z})\times(kz\hat{x})\,dz = 0$$

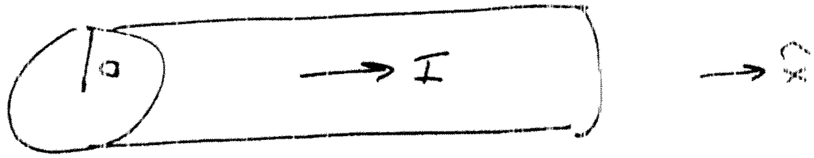
since intergration of odd function over even range.

$$= \vec{F}_{41}$$

$$\vec{F}_{34} = I(-a\hat{y})\times\left(\frac{a}{2}k\hat{x}\right) = \frac{Ia^2}{2}k\,\hat{z}$$

$$\vec{F}_{total} = \sum\vec{F} = \vec{F}_{12} + \vec{F}_{41} = kIa^2\,\hat{z}$$

5.5



(a) $\vec{K} = \frac{I}{2\pi a}\hat{x} = \frac{I}{\text{circumference}}\hat{x}$

(b) $\vec{J} = \frac{J_0}{s}\hat{x}$

$$I = \int J\,da = \int_0^{2\pi} d\phi \int_0^a \cancel{s}\,d\cancel{s}\left(\frac{J_0}{s}\right)$$

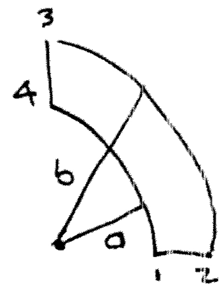
$$= 2\pi J_0 \int_0^a ds = 2\pi J_0 a$$

$$J_0 = \frac{I}{2\pi a}$$

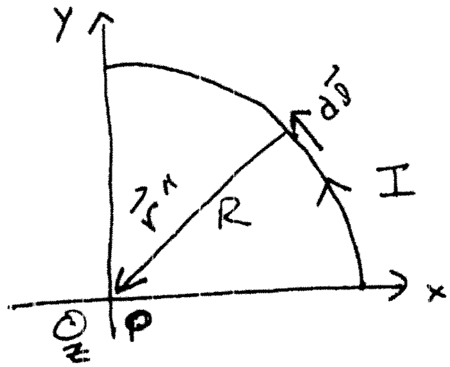
$$\vec{J} = \frac{J_0}{s}\hat{x} = \frac{I}{2\pi a s}\hat{x}$$

5.9

(a)



Magnetic field of quarter circle



Biot - Savart

$$\vec{B}_P = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \hat{r}''}{r''^2}$$

$$d\vec{l}' = R\,d\phi'\,\hat{\phi}' \qquad r'' = R$$

$$|d\vec{l}' \times \hat{r}''| = |d\vec{l}'|\,|\hat{r}''| \sin 90$$

$$= R\,d\phi'$$

$$d\vec{l} \times \hat{r}'' = R\,d\phi'\,\hat{z} \qquad (RHR)$$

$$\vec{B}_p = \frac{\mu_0 I}{4\pi} \frac{R\hat{z}}{R^2} \int_0^{\pi/2} d\phi'$$

$$= \frac{\mu_0 I}{8R} \hat{z}$$

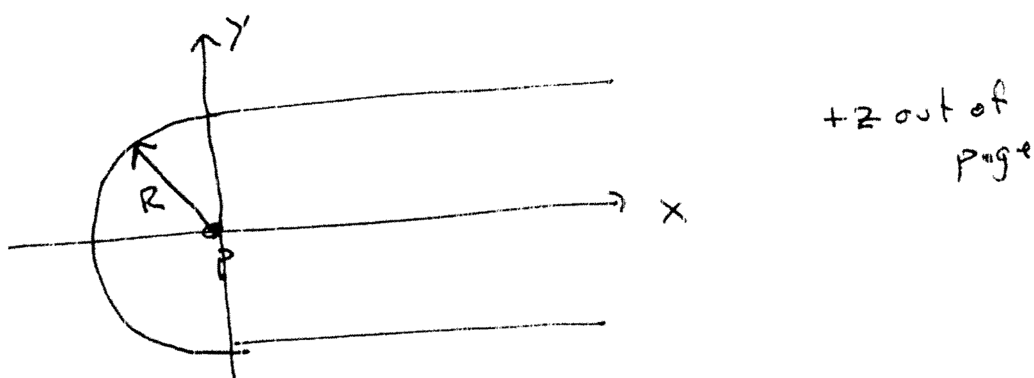
<u>Total Field</u>

$$\vec{B}_p = \vec{B}_{12} + \vec{B}_{23} + \vec{B}_{34} + \vec{B}_{41}$$

$$\quad\; = 0 - \frac{\mu_0 I}{8b}\hat{z} + 0 + \frac{\mu_0 I}{8a}\hat{z}$$

$$= \frac{\mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)$$

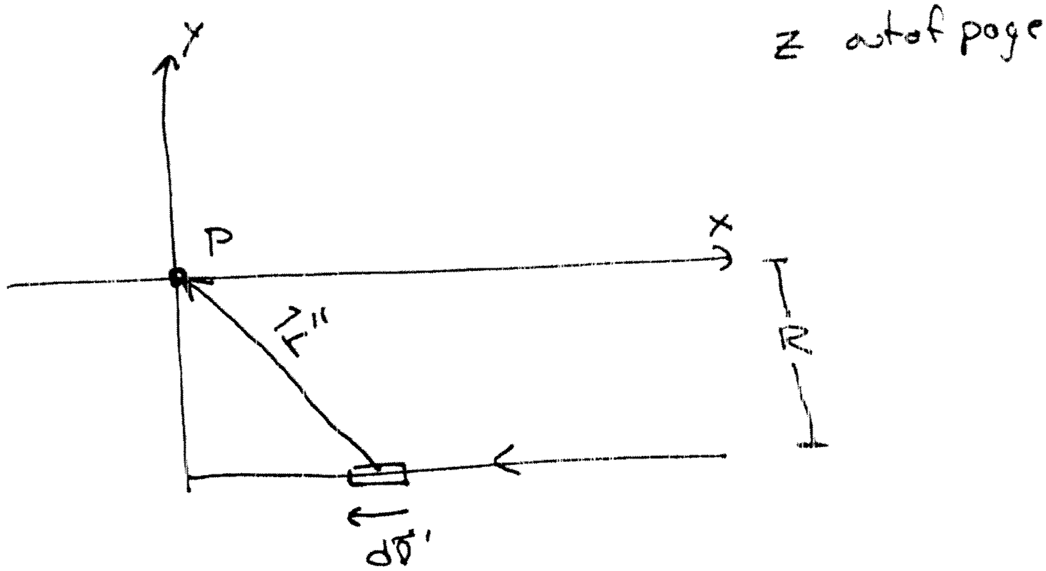
(b)



By similar calculation, the field of the half circle is

$$\frac{\mu_0 I}{4R}(-\hat{z})$$

The fields of the two lines are equal and must be calculated by integration.



$\vec{r}_P = (0, 0, 0)$ $\qquad d\vec{l}' = -dx'\hat{x}$ (integrating $0 \to \infty$)

$\vec{r}' = (x', -R, 0)$

$\vec{r}'' = \vec{r}_P - \vec{r}' = (-x', R, 0)$

$r'' = \sqrt{x'^2 + R^2}$

$$\vec{B}_P = \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{d\vec{l}' \times \vec{r}''}{r''^3}$$

$$d\vec{l}' \times \vec{r}'' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -dx' & 0 & 0 \\ -x' & R & 0 \end{vmatrix} = -dx' R \hat{z}$$

$$\vec{B}_p = -\frac{\mu_0 I R}{4\pi}\hat{z} \underbrace{\int_0^{\infty} \frac{dx'}{(x'^2+R^2)^{3/2}}}_{1/R^2}$$

$$= \frac{-\mu_0 I R}{4\pi R^2}\hat{z}$$

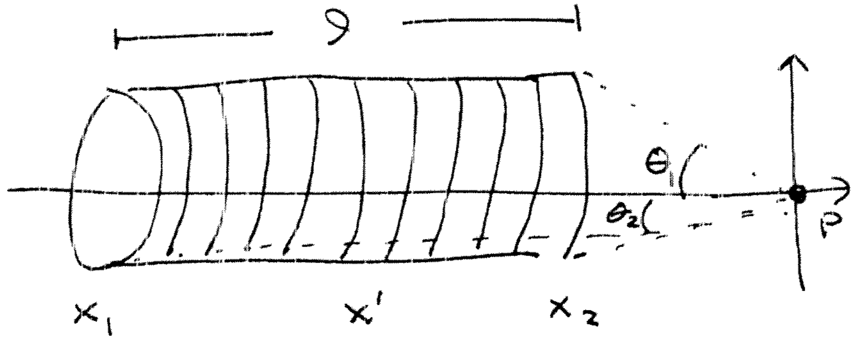
$$= \frac{-\mu_0 I}{4\pi R}\hat{z} \equiv \vec{B}_{wire,\, bottom} = \vec{B}_{wire,\, top}$$

$$\vec{B} = \vec{B}_{circle} + \vec{B}_{wire,\, bottom} + B_{wire,\, top}$$

$$= -\hat{z}\left(\frac{\mu_0 I}{4R} + 2\cdot\frac{\mu_0 I}{4\pi R}\right)$$

$$= -\hat{z}\,\frac{\mu_0 I}{4R}\left(1 + \frac{2}{\pi}\right)$$

5.11



Let x' be location of one of the rings

$$K = I\frac{N}{\ell}$$

$$dI = K\,dx' = I\frac{N}{\ell}\,dx'$$

$$\vec{B}_P = \int dB = \int_{x_1}^{x_2} \frac{\mu_0 dI}{2}\frac{a^2}{(a^2+x'^2)^{3/2}}\hat{x}$$

$$= \mu_0 \frac{N}{\ell}\frac{I}{2}a^2 \underbrace{\int_{x_1}^{x_2} \frac{dx'}{(a^2+x'^2)^{3/2}}}\ \hat{x}$$

$$= \frac{\mu_0 N I a^2 \hat{x}}{2\ell}\left[\frac{x}{a^2\sqrt{x^2+a^2}}\right]_{x_1}^{x_2}$$

$$= \frac{1}{2}\mu_0 \frac{N}{\ell} I \hat{x}\left(\frac{x_2}{\sqrt{x_2^2+a^2}} - \frac{x_1}{\sqrt{x_1^2+a^2}}\right)$$

$x_1 = -a\cos\theta_2 \qquad x_2 = -a\cos\theta_1$

$$\frac{x_1}{\sqrt{x_1^2+a^2}} = -\cos\theta_2$$

$$\frac{x_2}{\sqrt{x_2^2+a^2}} = -\cos\theta_1$$

$$\vec{B}_p = \frac{1}{2}\mu_0 \frac{N}{l} I\hat{x}\left(\cos\theta_2 - \cos\theta_1\right)$$

If solenoid begomes long, $\theta_1 \to \pi$, $\theta_2 \to 0$

$$(\ ) \to 2$$

$$\vec{B}_p = \mu_0 \frac{N}{l} I\hat{x}$$ ✓