Homework 8

Due Friday 4/5/2013 - at beginning of class

Griffiths' 4 Problems

5.17 (Griffiths' 3rd Edition Problem 5.16)

5.23 (Griffiths' 3rd Edition Problem 5.22)You only need to find the potential. You do not need to take the curl to find the field.

5.24 (Griffiths' 3rd Edition Problem 5.23)

5.37(a) (Griffiths' 3rd Edition Problem 5.35)

6.1

6.3 - Work part (b) only.

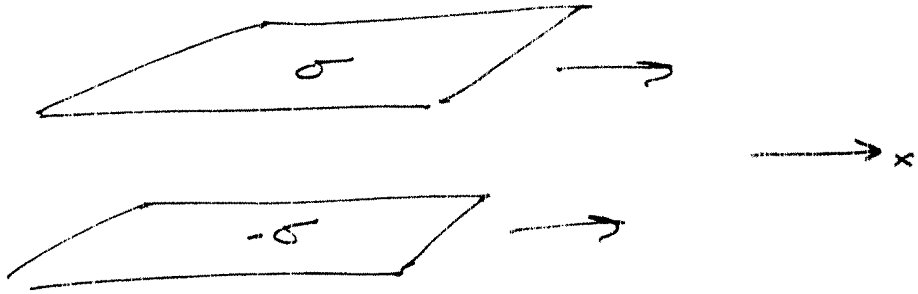
Additional Problems

E.8.1 A non-uniform current $\vec{J} = \gamma r^2 \hat{z}$ flows in the $\hat{z}$ direction in the region $a < s < b$. γ is a constant. Compute the magnetic field everywhere.

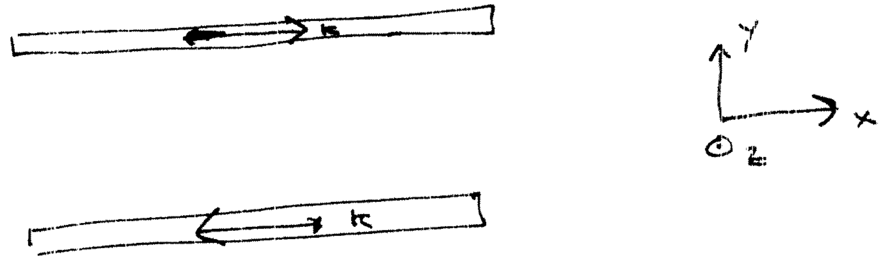
E.8.2 Compute the vector potential at the center of a square sheet of current $\vec{K} = K_0 \hat{y}$ where the current extends from $x = -a$ to a and $y = -a$ to a in the $x - y$ plane. K_0 is a constant.

E.8.3 A flat square loop of wire with side length ℓ in in the x-y plane centered at the origin. The loop carries a current I in the clockwise direction when viewed from the positive z axis. Compute the vector potential at a point a distance $R > \ell$ along the x axis.

5.17



$\vec{K}_t = \sigma v \hat{x}$ $\qquad$ $\vec{K}_b = -\sigma v \hat{x}$



The field of one plate is

$$\vec{B} = \begin{cases} \mu_0 \dfrac{K}{2}\,\hat{z} & \text{above} \\ -\mu_0 \dfrac{K}{2}\,\hat{z} & \text{below} \end{cases}$$

The fields of the two plates cancel above and below and add in the middle

$$\text{(a)} \qquad \vec{B}_{\text{two plates}} = \begin{cases} 0 & \text{above} \\ -\mu_0 K \hat{z} & \text{between} \\ 0 & \text{below} \end{cases}$$

The magnetic force per unit area on top plate is

$$P_m = \vec{K}_t \times \frac{1}{2}\left(\vec{B}_{above} + \vec{B}_{between}\right)$$

$$= (\sigma v \hat{x}) \times \left(-\frac{1}{2}\mu_0 K \hat{z}\right)$$

$$= \frac{1}{2}\mu_0 (\sigma v)^2 \hat{y} \quad \text{(upward on plate)}$$

(c) Electric Pressure

$$P_e = \sigma \frac{1}{2}\left(\underset{0}{\vec{E}_{above}} + \underset{\frac{\sigma}{\varepsilon_0}}{\vec{E}_{between}}\right)$$

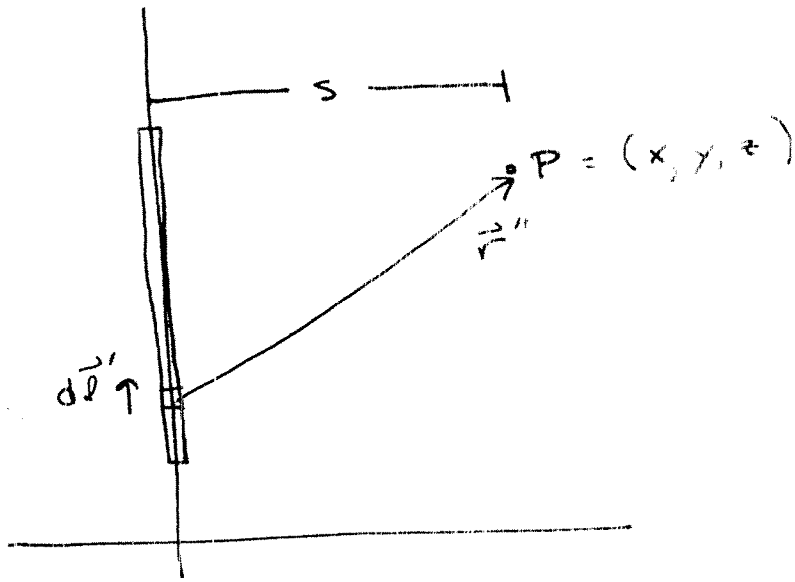
$$= -\frac{1}{2}\frac{\sigma^2}{\varepsilon_0}\hat{y}$$

The pressures are equal when

$$\frac{1}{2}\mu_0 \sigma^2 v^2 = \frac{1}{2}\frac{\sigma^2}{\varepsilon_0}$$

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c \quad \text{speed of light}$$

5.23



$$\vec{r}'' = s'\hat{s}' + (z - z')\hat{z}$$

$$r'' = \sqrt{s'^2 + (z - z')^2}\ \hat{z} \qquad d\vec{l}' = dz'\hat{z}$$

$$\vec{A} = \frac{\mu_0 I \hat{z}}{4\pi} \int_{z_1}^{z_2} \frac{dz'}{\sqrt{s^2 + (z - z')^2}} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}'}{r''}$$

$$= \frac{\mu_0 I \hat{z}}{4\pi} \Big[\ln\big(z' - z + \sqrt{s^2 + (z' - z)^2}\,\Big]_{z_1}^{z_2}$$

$$= \frac{\mu_0 I \hat{z}}{4\pi} \ln\left(\frac{z_2 - z + \sqrt{s^2 + (z_2 - z)^2}}{z_1 - z + \sqrt{s^2 + (z_1 - z)^2}} \right)$$

5.24

$$\vec{A} = k\hat{\phi}$$

$$\nabla \times \vec{A} = \underbrace{-\frac{\partial A_\phi}{\partial z}}_{=0}\hat{s} + \underbrace{\frac{1}{s}\frac{\partial}{\partial s}(sA_\phi)}_{=\frac{k}{s}}\hat{z}$$

$$\vec{B} = \frac{k}{s}\hat{z}$$

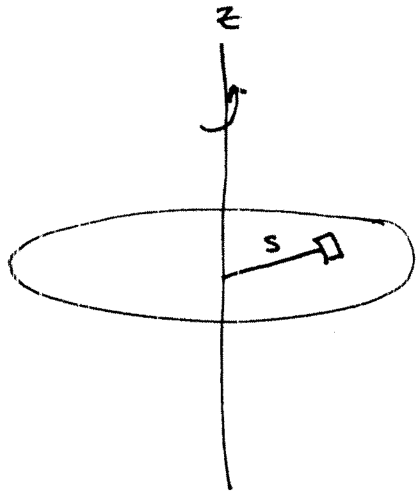
<u>Ampere's Law</u>

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{J} = \frac{1}{\mu_0}\nabla \times \vec{B} = \frac{1}{\mu_0}\left(\underbrace{\frac{1}{s}\frac{\partial B_z}{\partial \phi}}_{=0}\hat{s} - \underbrace{\frac{\partial B_z}{\partial s}}_{=+\frac{k}{s^2}}\hat{\phi}\right)$$

$$\vec{J} = \frac{+k}{\mu_0 s^2}\hat{\phi}$$

5.37 (a)



$$\vec{K} = s\omega\sigma\,\hat{\phi}$$

$$d\vec{I} = \vec{K}\,ds$$

$$d\vec{m} = |d\vec{I}|\,A\,\hat{z}$$

$$A = \pi s^2$$

$$\vec{m} = \int d\vec{m} = \int_0^R (s\omega\sigma)(\pi s^2)\,ds$$

$$= \pi\omega\sigma \int_0^R s^3\,ds$$

$$= \frac{\pi\omega\sigma}{4} R^4$$

6.1

The loop will rotate until its moment points to the bottom of the page

The moment of the round loop is

$$\vec{m}_1 = \pi a^2 I \hat{y}$$

The field at loop 2 is

$$\vec{B}_{12} = -\frac{\mu_0}{4\pi}\frac{m}{x^3}\hat{y} = -\frac{\mu_0}{4\pi}\frac{\pi a^2 I}{x^3}$$

The moment of loop 2 is

$$\vec{m}_2 = b^2 I \hat{x}$$

The torque on loop 2

$$\vec{T}_{12} = \vec{m}_2 \times \vec{B}_{12}$$
$$= (b^2 I)\left(\frac{\mu_0}{4\pi}\frac{\pi a^2 I}{r^3}\right)(-\hat{z})$$

$$\vec{\tau}_{12} = -\frac{a^2 b^2 I^2}{4r^3}\hat{z}$$

6.3

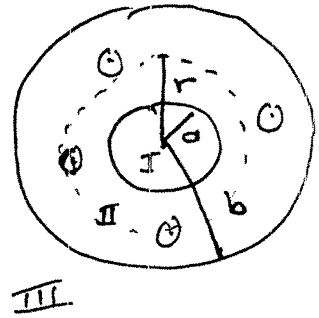
$$B_{12} = \frac{2\mu_0}{4\pi}\frac{m_1}{r^3}$$

$$F_{12} = \nabla\left(\vec{m}_2 \cdot \vec{B}_{12}\right)$$

$$= \nabla\left(\frac{\mu_0}{2\pi}\,\frac{m_1 m_2}{r^3}\right)$$

$$\vec{F}_{12} = -\frac{3}{2}\,\frac{\mu_0}{\pi}\,\frac{m_1 m_2}{r^4}\,\hat{x}$$

E1



<u>Region I</u> $r<a$ $\quad I_{enc} = 0 \quad \vec{B}_I = 0$

<u>Region II</u> $a<r<b$

$$I_{enc} = \int \vec{J}\cdot\hat{n}\, da$$

$$da = (dr)(r\,d\phi) = r\,dr\,d\phi$$

$$\hat{n} = \hat{z}$$

$$= \int_a^r dr \int_0^{2\pi} d\phi\, r\, J$$

$$= \int_a^r dr \int_0^{2\pi} \gamma r^3\, d\phi$$

$$= \frac{\gamma r^4}{4}\Big|_a^r = \frac{\gamma}{4}(r^4 - a^4)$$

<u>Ampere's Law</u>

$$\oint \vec{B}\cdot d\vec{l} = 2\pi r B = \mu_0 I_{enc}$$

$$\vec{B} = \frac{\mu_0 I_{enc}}{2\pi r}$$ counterclockwise (from RHR)

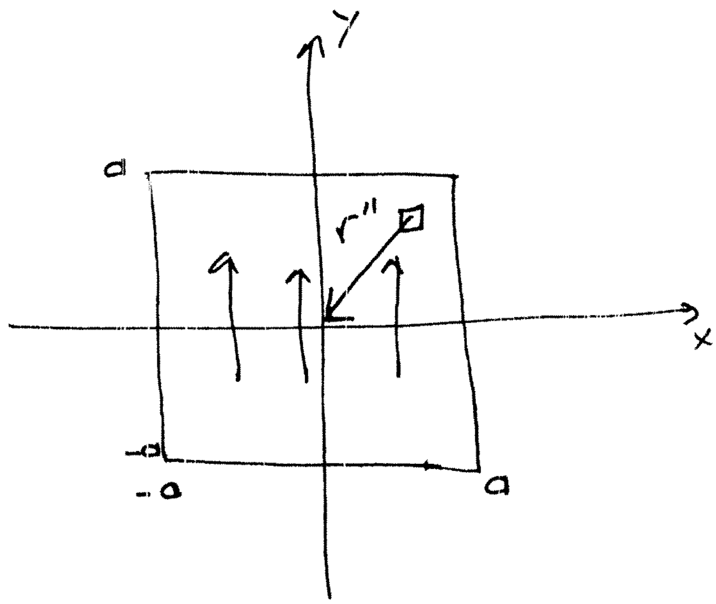
$$\vec{B}_{II} = \frac{\mu_0 \gamma}{8\pi r} \cdot (r^4 - a^4)$$

Region III

$$I_{enc} = \frac{\gamma}{4}(b^4 - a^4)$$

$$\vec{B}_{III} = \frac{\mu_0 I_{enc}}{2\pi r} = \frac{\mu_0 \gamma}{8\pi r}(b^4 - a^4)$$

E2



$$\vec{K} = K_0 \hat{y}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}\, da'}{r''}$$

$$\vec{r} = (0,0,0) \qquad \vec{r}\,' = (x', y', 0)$$

$$r'' = \sqrt{x'^2 + y'^2}$$

$$da' = dx'\, dy'$$

$$\vec{A} = \frac{\mu_0 K_0}{4\pi} \hat{y} \int_{-a}^{a} dy' \int_{-a}^{a} dx' \frac{1}{\sqrt{x'^2 + y'^2}}$$

$$\vec{A} = \frac{\mu_0 K_0 \hat{y}}{4\pi} \left(4a \ln\left(\frac{1+\sqrt{2}}{\sqrt{2}-1}\right)\right)$$

$$\vec{A} = \frac{\mu_0 k_0 a}{\pi} \hat{y} \ln\left(\frac{1+\sqrt{2}}{\sqrt{2}-1}\right)$$

> $assume(a,\ positive);$

> $int\left(int\left(\frac{1}{\left(x^2 - y^2\right)^{\left(\frac{1}{2}\right)}},\ x = -a..a\right),\ y = -a..a\right);$

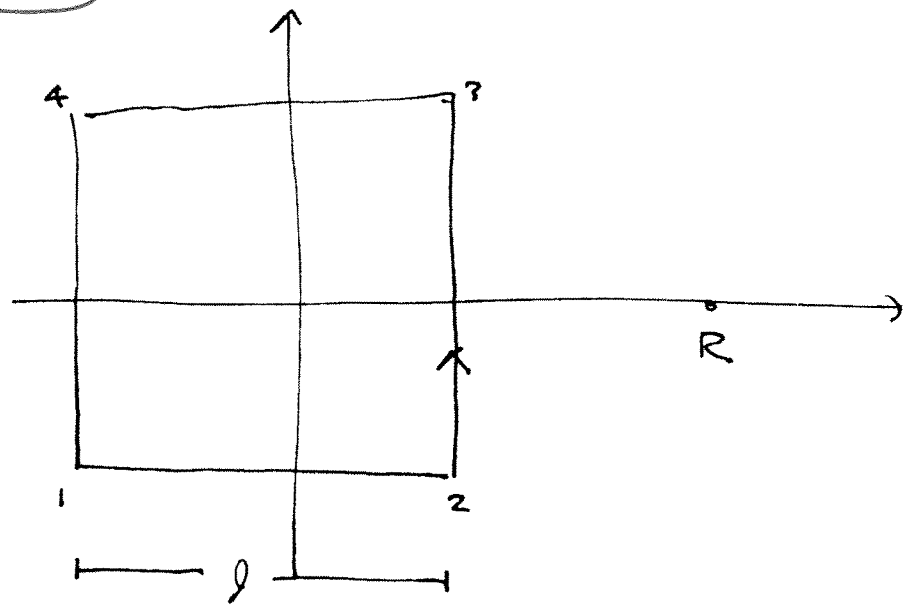
$$-4\,a\sim \ln\left(\sqrt{2}-1\right) + 4\,a\sim \ln\left(1+\sqrt{2}\right) \tag{1}$$

> $simplify(\%)$

$$-4\,a\sim \left(\ln\left(\sqrt{2}-1\right) - \ln\left(1+\sqrt{2}\right)\right) \tag{2}$$

>

E.8.3



$$\vec{A} = \vec{A}_{12} + \vec{A}_{23} + \vec{A}_{34} + \vec{A}_{41}$$

$$\vec{A}_{12} + \vec{A}_{34} = 0 \quad \text{by symmetry}$$

$$\vec{A}_{12} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}}{r''}$$

$$r'' = \sqrt{y^2 + (R - l/2)^2}$$

Let $R - l/2 = d_{12R}$

$$= \frac{\mu_0 I}{4\pi}\hat{y}\int_{-l/2}^{l/2} \frac{dy}{\sqrt{y^2 + d_{12R}^2}}$$

$$= \frac{\mu_0 I}{2\pi}\hat{y}\int_{0}^{l/2} \frac{dy}{\sqrt{y^2 + d_{12R}^2}}$$

$$\int \frac{dy}{\sqrt{y^2+a^2}} = \ln\left(y + \sqrt{y^2+a^2}\right) \quad \text{Schaum's}$$

$$\vec{A}_{12} = \frac{\mu_0 I}{2\pi}\hat{y} \ln\left(\frac{l/2 + \sqrt{(l/2)^2 + d_{12R}^2}}{d_{12R}}\right)$$

$$\vec{A}_{41} = -\frac{\mu_0 I}{2\pi}\hat{y} \ln\left(\frac{l/2 + \sqrt{(l/2)^2 + d_{41R}^2}}{d_{41R}}\right)$$

$$d_{41R} = R + l/2$$

$$\vec{A} = \vec{A}_{12} + \vec{A}_{41}$$

$$= \frac{\mu_0 I}{4\pi}\hat{y} \ln\left(\left(\frac{R+l/2}{R-l/2}\right)\frac{l/2 + \sqrt{(l/2)^2 + (R-l/2)^2}}{l/2 + \sqrt{(l/2)^2 + (R+l/2)^2}}\right)$$