Homework 9

Due Friday 4/12/2013 - at beginning of class

Griffiths' 4 Problems (3rd Edition numbers are the same)

6.8

6.12

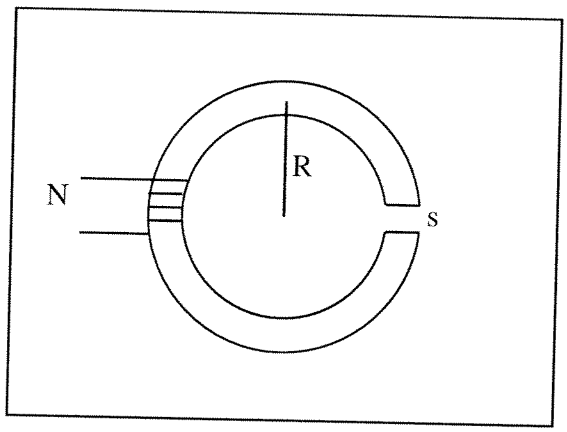
6.15

6.17

6.18

Additional Problems

E.9.1 Compute the magnetic field in the gap ($s = 0.15$cm) in a toroidal iron ring. The radius of the ring is $R = 8.1$cm. The ring is wrapped with 100 turns carrying current $I = 0.7$A. At the operating current, the relative permeability of the ring is $\mu_r = 100$.



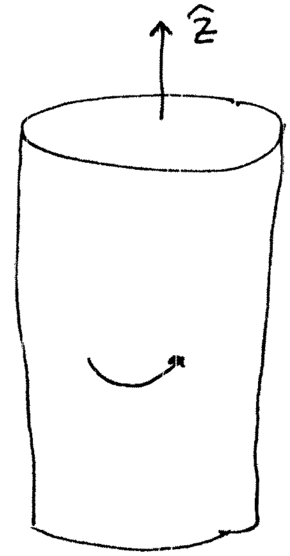
E.9.2 Repeat problem E2 where 1/10 the circumference of the iron is replaced with a permanent magnetic material with magnetization $M = 10^5$A/m. The permanent magnet produces a field in the same direction as the field produced by the wraps of wire.

E.9.3 A manufacturer of Alnico magnets reports a residual field of 12,500 Gauss. This is the field in the center of an infinitely long magnet. Compute the magnetization $\vec{M}$. Our cylindrical Alnico lab magnets were about 10cm long with radius 0.5cm. Compare the field at the center of the flat surface of this magnet to field at the surface of a disk magnet of height 1mm and radius 0.5cm. You may use the finite solenoid formula. Also compute the field of the disk magnet modelling the bound current as a ring of current. Modelling the two magnets as point dipoles, compare the magnetic field at 30cm in the direction of the moment of the two magnets.

E.9.4 A circular magnet with radius $a = 1$cm and thickness $d = 1$mm and magnetization 1×10^5A/m lies in the $x - y$ plane centered at the origin.

- **a** Calculate the magnetic field at the center of the magnet.
- **b** Calculate the torque a magnetic field $\vec{B} = B_0\hat{x}$ would exert on the magnet if $B_0 = 0.2$T.

(6.8)



$\vec{M} = k s^2 \hat{\phi}$

The problem has azimuthal symmetry, so the magnetic field must be circular about the axis. $\Rightarrow$ $\vec{B} = B\hat{\phi}$

There are no free currents, so

$$\oint \vec{H} \cdot d\vec{l} = 0$$

$\Rightarrow$ $\vec{H} = 0$ everywhere.

<u>Outside</u> $\vec{M}_o = 0$ $\mu_0 H = 0$ $\Longrightarrow$ $\vec{B}_o = 0$

<u>Inside</u> $\mu_0 \vec{H} + \mu_0 \vec{M} = \vec{B}_i$ $\Rightarrow$ $\vec{B}_i = \mu_0 k s^2 \hat{\phi}$

As a check, we can also attack this through the ~~__~~ bound currents.

$$\vec{J}_b = \nabla \times \vec{M}$$

$$= -\frac{\partial M_\phi}{\partial z}\hat{s} + \frac{1}{s}\frac{\partial}{\partial s} s M_\phi \hat{z}$$

$$= \frac{1}{s}\frac{\partial}{\partial s} k s^3 \hat{z}$$

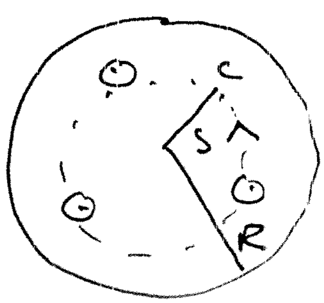
$$= 3ks\hat{z}$$

Surface Current

$$\vec{K}_b = \vec{M} \times \hat{s} = ks^2 \hat{\phi} \times \hat{s}$$

$$= -ks^2\hat{z} \quad \text{since } \hat{s} \times \hat{\phi} = \hat{z}$$

End View



Inside $\quad I_{enc} = \int_0^s J_b \, da \qquad da = ds \, s \, d\phi$

$$= \int_0^s ds \int_0^{2\pi} d\phi \, s \, J_b$$

$$= \int_0^s ds \int_0^{2\pi} d\phi \; 3ks^2$$

$$= 6\pi \int_0^s ds \, ks^2$$

$$= 2\pi k s^3$$

Field inside

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = 2\pi s \, B_i$$

$$B_i = \mu_0 \frac{I_{enc}}{2\pi s} = \frac{\mu_0 \, 2\pi k s^3}{2\pi s} = \mu_0 k s^2$$

Outside $\quad I_{enc} = 2\pi k R^3 + K_b \cdot 2\pi R$

$$= 0$$

$$\vec{B}_o = 0$$

6.12

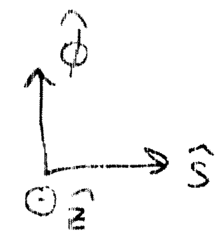
$$\vec{M} = ks\,\hat{z}$$

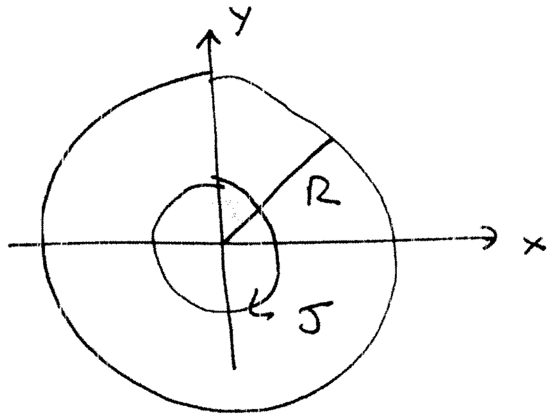
(a) $$\vec{J}_b = \nabla \times \vec{M}$$

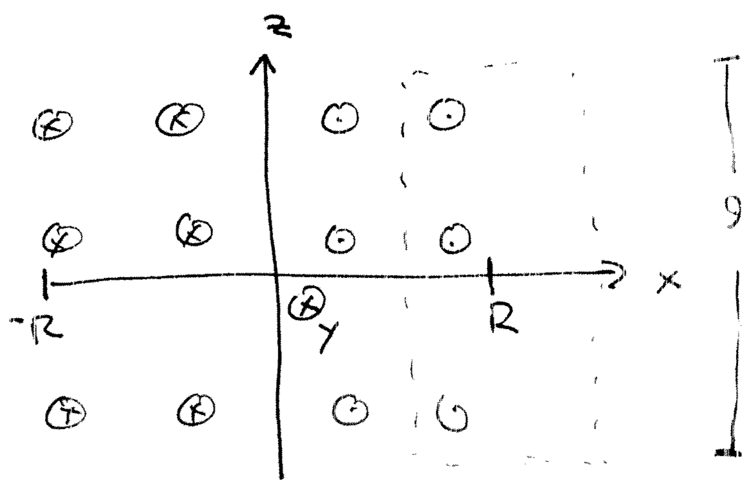
$$= \frac{1}{s}\frac{\partial M_z}{\partial \phi}\,\hat{s} - \frac{\partial M_z}{\partial s}\,\hat{\phi}$$

$$= -k\hat{\phi}$$

$$\vec{K}_b = \vec{M} \times \hat{s} = ks\,\hat{z} \times \hat{s} = kR\hat{\phi}$$







Use an Amperian Path of length l as drawn.

The system is a concentric stack of infinite solenoids so the field outside is zero using the same reasoning as the infinite solenoid.

Inside the current encircled by the Amperian path is

$$I_{enc} = l\int_s^R J_b\, ds + l K b$$

$$= l(-k)\int_s^R ds + l k R$$

$$= l(-k)(R-s) + l k R = l k s$$

<u>Ampere's Law</u>

$$\oint \vec{B}\cdot d\vec{l} = Bl = \mu_0 I_{enc}$$

$$\vec{B}_i = \mu_0 k s\, \hat{z}$$

* Note, direction is subtle, always more surface current than volume current encircled.

(b) No free currents + azimuthal symmetry

$$\oint \vec{H} \cdot d\vec{l} = 2\pi s H = 0$$

$\vec{H} = 0$ everywhere

$$\vec{B}_o = \mu_0 \vec{H} = 0 \qquad \text{Outside}$$

$$\vec{B}_i = \mu_0 \vec{H} + \mu_0 \vec{M} = \mu_0 k s \hat{z}$$

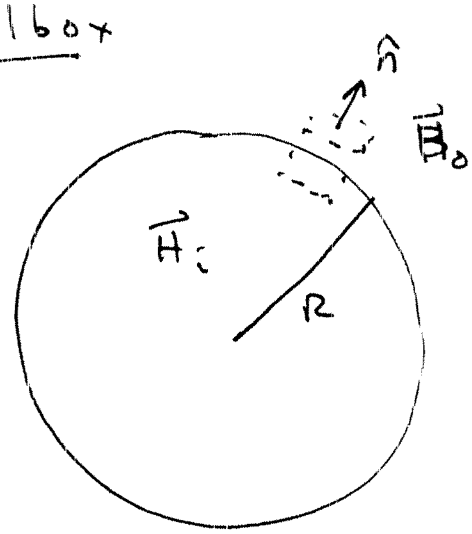
(6.15) Uniformly magnetized sphere $\vec{M} = M_0 \hat{z}$

Let $\vec{H} = -\nabla W$, since $\nabla \times \vec{H} = 0$ except at surface of sphere.

$$\nabla \cdot \vec{B} = 0 = \mu_0 (\nabla \cdot \vec{H} - \nabla \cdot \vec{M})$$

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$$

Gaussian pillbox



$$\vec{H}_0 \cdot \hat{n} - \vec{H}_i \cdot \hat{n} = \vec{M}_i \cdot \hat{n} - \underset{\underset{0}{\shortparallel}}{\vec{M}_0} \cdot \hat{n} = \vec{M} \cdot \hat{n}$$

$$-\nabla W_0 \cdot \hat{n} + \nabla W_i \cdot \hat{n} = \vec{M} \cdot \hat{n}$$

$$-\frac{\partial W_o}{\partial r}\Big|_R + \frac{\partial W_i}{\partial r}\Big|_R = \vec{M}\cdot\hat{n} = M_o \cos\theta$$

We also have

$$W_i(R,\theta) = W_o(R,\theta) \quad \text{from} \quad \vec{H} = -\nabla W.$$

As we have done before, inside and outside Laplace's equation is satisfied, so discarding explosive terms

$$W_i = \sum A_n r^n P_n(\cos\theta)$$

$$W_o = \sum B_n r^{-(n+1)} P_n(\cos\theta)$$

<u>Continuity</u>

$$W_i(R,\theta) = \sum A_n R^n P_n(\cos\theta) =$$

$$\sum B_n R^{-(n+1)} P_n(\cos\theta)$$

Orthogonality

$$A_n R^n = B_n / R^{n+1} \Rightarrow B_n = R^{2n+1} A_n$$

ES Boundary Condition

$$-\left.\frac{\partial W_o}{\partial r}\right|_R + \left.\frac{\partial W_i}{\partial r}\right|_R = M_0 P_1(\cos\theta) = M_0 \cos\theta$$

$$-\sum_n -(n+1) B_n R^{-(n+2)} P_n(\cos\theta)$$

$$+ \sum n A_n R^{n-1} P_n(\cos\theta) = M_0 P_1(\cos\theta)$$

$$\sum_n (2n+1) A_n R^{n-1} P_n(\cos\theta) = M_0 P_1(\cos\theta)$$

Orthogonality

$$A_n = 0 \Rightarrow B_n = 0 \quad \text{for } n \neq 1$$

$n=1$ $$(2\cdot 1+1) A_1 R^{1-1} = M_0$$

$$A_1 = \frac{M_0}{3}$$

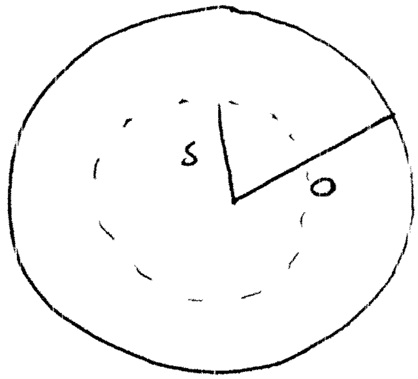
$$W_i = \frac{M_0}{3} r\cos\theta = \frac{M_0}{3} z$$

$$\vec{H}_i = -\nabla W_i = -\frac{M_0}{3}\hat{z}$$

$$\vec{B}_i = \mu_0 \left(\vec{H}_i + \vec{M}_i \right) = \mu_0 \left(-\frac{M_0}{3}\hat{z} + M_0 \hat{z} \right)$$

$$= \frac{2}{3} \mu_0 M_0 \hat{z}$$

(6.17)



Current out of page $\hat{z}$

Field CCW $(+\hat{\phi})$

Free current encircled by Amperian path of radius $s < a$.

$$I_{fenc} = \int J\,da = \int_0^s ds \int_0^{2\pi} s\,d\phi\, J$$

$$J = \frac{I}{\pi a^2}$$

$$I_{fenc} = \frac{2\pi I}{\pi a^2} \int_0^s s\,ds = \frac{2\pi I}{\pi a^2}\,\frac{s^2}{2}$$

$$= I\, s^2/a^2$$

<u>Ampere's Law</u>

$$\oint \vec{H} \cdot d\vec{l} = I_{fenc} = 2\pi s H$$

Inside $s < a$

$$H_i = \frac{I_{fenc}}{2\pi s} = \frac{I\, s^2/a^2}{2\pi s}$$

$$= \frac{I s}{2\pi a^2} = B_i/\mu_r\mu_0$$

$$B_i = \frac{\mu_r \mu_0 I s}{2\pi a^2} = \frac{(1+\chi_m)\mu_0 I s}{2\pi a^2}$$

Outside $s > a$ $I_{fenc} = I$

$$H_o = \frac{I}{2\pi s} \qquad B_o = \mu_0 H_o$$

$$B_o = \frac{\mu_0 I}{2\pi s}$$

Magnetization

$$\vec{M} = \chi_m \vec{H}$$

$$\vec{M}_i = \frac{\chi_m I s}{2\pi a^2} \hat{\phi}$$

Volume Bound Current

$$\vec{J}_b = \nabla \times \vec{M}$$

$$= + \frac{1}{s} \frac{\partial}{\partial s} s M_{i\phi} \hat{z}$$

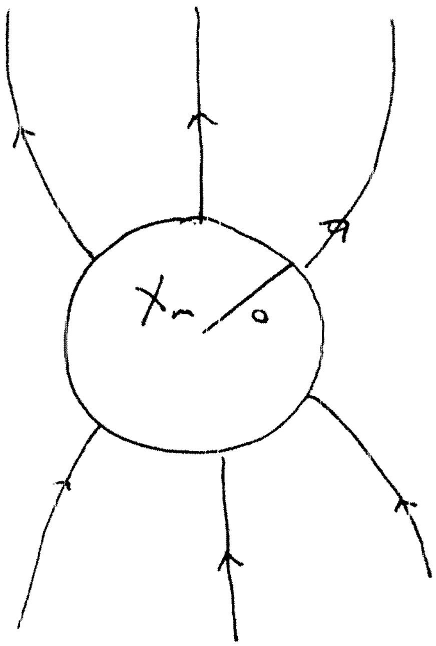
$$= + \frac{\chi_m I}{\pi a^2} \hat{z} = \chi_m \vec{J}_f$$

Surface Current

$$\vec{K}_b = \vec{M} \times \hat{s} = \frac{\chi_m I a}{2\pi a^2} \cdot (\hat{\phi} \times \hat{s})$$

$$= - \frac{\chi_m I}{2\pi a} \hat{z}$$

6.18



$\vec{B}_\infty = B_0 \hat{z}$

$\vec{H}$ satisfies Laplace's eqn except at boundary.

$$\vec{H} \equiv -\nabla W$$

Boundary Conditions

$$W_i(a,\theta) = W_o(a,\theta) \qquad \text{continuity}$$

$$\vec{H}_o \to \frac{\vec{B}_0}{\mu_0} = \frac{B_0}{\mu_0}\hat{z}$$

$$\to \quad W_o \Rightarrow -\frac{B_0}{\mu_0} z \qquad \text{as} \quad z \to \infty$$

$$= -\frac{B_0}{\mu_0} r P_1(\cos\theta)$$

$$\nabla \cdot \vec{B} = 0 \implies \vec{B}_o \cdot \hat{n} = \vec{B}_i \cdot \hat{n}$$

$$\vec{B}_o = \mu_o \vec{H}_o \qquad \vec{B}_i = \mu_o \mu_r \vec{H}_i$$

$$\vec{H}_o \cdot \hat{n} = \mu_r \vec{H}_i \cdot \hat{n}$$

$$\left.\frac{\partial W_o}{\partial r}\right|_a = \mu_r \left.\frac{\partial W_i}{\partial r}\right|_a$$

<u>Magnetic Potential</u>

$$W_i = \sum A_n r^n P_n(\cos\theta) \qquad \text{inside}$$

$$W_o = -\frac{B_o}{\mu_o} r P_1(\cos\theta) + \sum B_n r^{-(n+1)} P_n(\cos\theta)$$

Clearly, only $n=1$ term will survive,

$$W_1 = A_1 r P_1(\cos\theta)$$

$$W_o = -\frac{B_o}{\mu_o} r P_1(\cos\theta) + \frac{C_1}{r^2} P_1(\cos\theta)$$

Continuity

$$W_i(a,\theta) = W_o(a,\theta)$$

$$A_1 a = -\frac{B_o}{\mu_o} a + \frac{C_1}{a^2}$$

ES BC

$$\left.\frac{\partial W_o}{\partial r}\right|_a = -\frac{B_o}{\mu_o} P_1 - \frac{2C_1}{a^3} P_1$$

$$= \mu_r A_1 P_1 = \mu_r \left.\frac{\partial W_i}{\partial r}\right|_a$$

$$C_1 = A_1 a^3 + \frac{B_o}{\mu_o} a^3$$

$$2C_1 = -\frac{B_o}{\mu_o} a^3 - \mu_r A_1 a^3$$

$$3C_1 = A_1 (1-\mu_r) a^3$$

$\cancel{A} _r =$

$$2C_1 + \mu_r C_1 = -\frac{B_0}{\mu_0} a^3 + \mu_r \frac{B_0}{\mu_0} a^3$$

$$C_1 = \frac{B_0}{\mu_0} a^3 \left(\frac{\mu_r - 1}{\mu_r + 2} \right)$$

$$A_1 = \frac{3C_1}{(1-\mu_r)a^3} = -\frac{B_0}{\mu_0} \frac{3}{\mu_r + 2}$$

Potentials

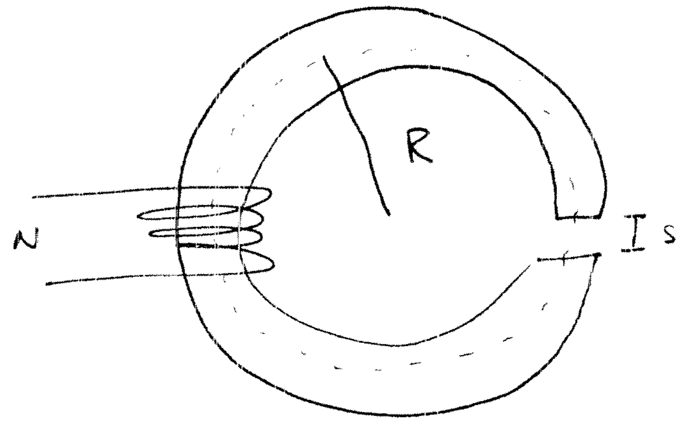
Inside

$$W_i = A_1 r \cos\theta = -\frac{B_0}{\mu_0} \frac{3}{\mu_r+2} z$$

$$\vec{H}_i = -\nabla W_i = \frac{B_0}{\mu_0} \frac{3}{\mu_r+2} \hat{z}$$

$$\vec{B}_i = \mu_r \mu_0 \vec{H}_i = \mu_r B_0 \frac{3}{\mu_r+2} \hat{z}$$

E1



$$\oint \vec{H} \cdot d\vec{l} = NI$$

$$= (2\pi R - s) H_i + s H_o$$

Since $\nabla \cdot \vec{B} = 0$ $\quad B_o = B_i \equiv B$

$$H_o = B/\mu_o \qquad B_i = \mu_o H_i + \mu_o M$$

$$H_i = B/\mu_o \mu_r$$

$$NI = (2\pi R - s) \frac{B}{\mu_o \mu_r} + s \frac{B}{\mu_o}$$

$$\mu_o \mu_r NI = (2\pi R - s) B + B \mu_r s$$

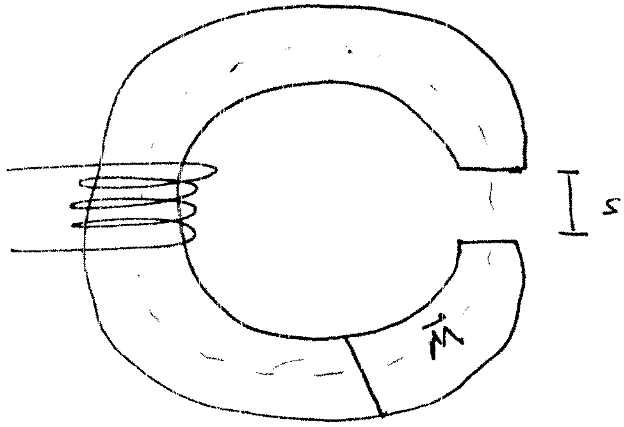
$$= B (2\pi R - s + \mu_r s)$$

$$B = \frac{\mu_0 \mu_r N I}{2\pi R - s + \mu_r s}$$

$$= \frac{\left(4\pi \times 10^{-7} \frac{\text{Tm}}{\text{A}}\right)(100)(100)(0.7\text{A})}{2\pi(0.081\text{m}) - 0.0015\text{m} + (100)(0.0015\text{m})}$$

$$= 0.013 \text{ T}$$

(E2)



H_0 = Field in Gap

H_i = Field in Iron

H_m = Field in magnetic

$\nabla \cdot \vec{B} = 0 \Rightarrow$ Magnetic field in all are equal.

$H_0 = B/\mu_0 \qquad H_i = \dfrac{B}{\mu_r \mu_0}$

$$B = \mu_0 H_m + \mu_0 M \Rightarrow H_m = \frac{B - \mu_0 M}{\mu_0}$$

Recall, H_m points in opposite direction to B.

$$\text{mmf} = \oint \vec{H} \cdot d\vec{l} = NI = s H_0 + \frac{1}{10} \cdot (2\pi R - s) H_m$$

$$+ \frac{9}{10} (2\pi R - s) H_i = NI$$

$$s\frac{B}{\mu_0} + \frac{1}{10}(2\pi R - s)\left(\frac{B - \mu_0 M}{\mu_0}\right)$$

$$+\frac{9}{10}(2\pi R - s)\frac{B}{\mu_0 \mu_r} = NI$$

$$10 s \mu_r B + \mu_r (2\pi R - s)(B - \mu_0 M)$$

$$+ 9(2\pi R - s) B = \mu_r \mu_0 N I \cdot 10$$

$$10 s \mu_r B + \mu_r (2\pi R - s) B + 9 (2\pi R - s) B$$

$$= 10 \mu_r \mu_0 N I + \mu_0 \mu_r M (2\pi R - s)$$

$$B\left(10 s \mu_r + (\mu_r + 9)(2\pi R - s)\right) =$$

$$B = \frac{\mu_0 \mu_r \left(10 N I + M(2\pi R - s)\right)}{10 s \mu_r + (\mu_r + 9)(2\pi R - s)}$$

$$= \frac{\mu_0 \cdot 100 \left(10 \cdot 100 \cdot 0.7A + 10^5 \frac{A}{m} \cdot (2\pi (0.081m) - 0.0015m)\right)}{10 \cdot 0.0015m \cdot 100 + (109)(2\pi (0.081m) - 0.0015m)}$$

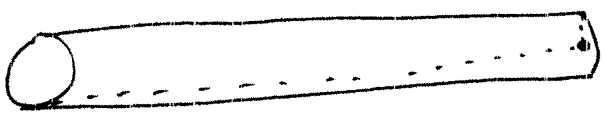
$$= \frac{\mu_0 \cdot 100 \cdot (700A + 50743A)}{1.5m + 55.3m} = 0.14T$$

(E3) $B_r = 1.25\ T$

Infinitely long magnet, with no free currents

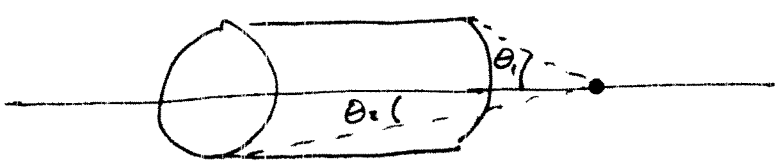
$$\vec{H} = 0 \qquad \vec{B} = \mu_0 \vec{M}$$

$$M = \frac{B_r}{\mu_0} = 9.9 \times 10^5\ A/m$$



Surface current $\quad K_b = |\vec{M} \times \hat{n}| = 9.9 \times 10^5\ A/m$

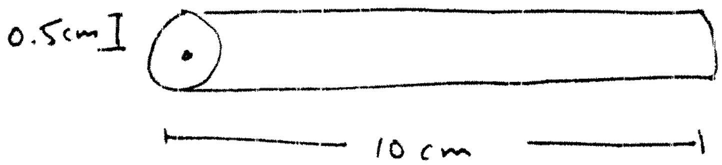
Finite Solenoid Formula



$$B = \frac{\mu_0 K_b}{2} \left(\cos\theta_2 - \cos\theta_1 \right)$$

$$B = 0.625\,T \left(\cos\theta_2 - \cos\theta_1 \right)$$

Long Magnet



$\theta_1 = \frac{\pi}{2} \qquad \theta_2 = \tan^{-1}\left(\frac{0.5\,cm}{10\,cm}\right) = 2.86^\circ$

$B = (0.625\,T)(\cos 2.86^\circ - \cos 90^\circ)$

$= 0.624\ T$

Disk Magnet

$\theta_1 = \pi/2$

$\theta_2 = \tan^{-1}\left(\frac{0.5\,cm}{0.1\,cm}\right)$

$= 78.7^\circ$

$B = (0.625\,T)(\cos 78.7^\circ)$

$= 0.123\,T$

Long Magnet Moment

$$m = MV = M\pi r^2 h$$

$$= (9.9\times10^5 A/m)\,\pi\,(0.005m)^2(0.1m)$$

$$= 7.78\ Am^2$$

Field

$$B = \frac{\mu_0}{4\pi}\frac{m}{r^3} = \frac{\left(1\times10^{-7}\frac{Tm}{A}\right)(7.78\ Am^2)}{(0.3m)^3}$$

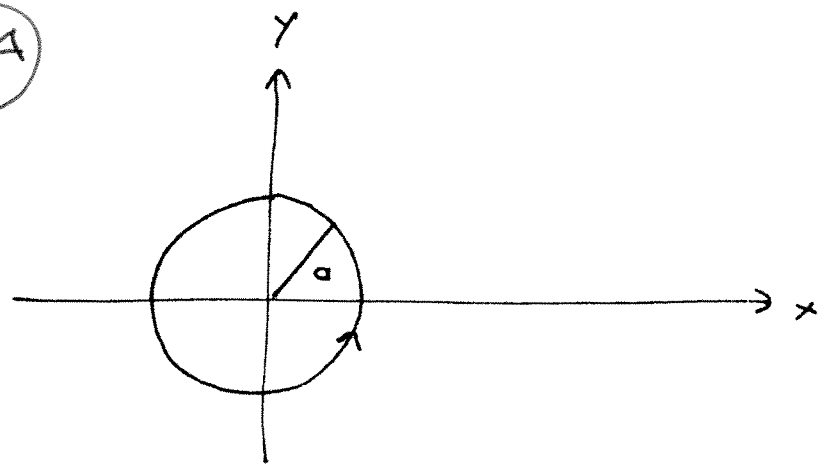
$$= 2.9\times10^{-5}\ T$$

Disk Magnet

$$m = M\pi r^2 h = 0.0778\ Am^2$$

$$B = 2.9\times10^{-7}\ T$$

E.9.4



The bound current density $\vec{K}_b = \vec{M} \times \hat{n} = M_0 \hat{\phi}$

The total bound current is $I = K_b d = M_o d = 100A$.

The field at the center is found by integrating the current around the loop

$$\vec{B} = \oint_c d\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \hat{r}}{a^2}$$

$d\vec{l} = a\,d\phi\,\hat{\phi}$

$\hat{r} = -\hat{r}$

$\hat{\phi} \times -\hat{r} = \hat{z}$

$$= \frac{\mu_0 I}{4\pi a^2} \hat{z} \int_0^{2\pi} a\, d\phi$$

$$= \frac{\mu_0 I}{4\pi a} \cdot 2\pi \hat{z} = \frac{\mu_0 I}{2a} \hat{z} = \frac{\mu_0 M_o d}{2a} \hat{z}$$

$$= \frac{\left(4\pi \times 10^{-7} \frac{Tm}{A}\right)(100A)}{2(0.01m)} \hat{z} = 2\pi \times 10^{-3}\, T\, \hat{z}$$

(b) The moment is $\vec{m} = M_0 V \hat{z} = M_0 \pi a^2 l \hat{z}$

where I used the right-hand rule for the moment to get the direction.

$$\vec{\tau} = \vec{m} \times \vec{B} = B_0 M_0 \pi a^2 l \; \hat{z} \times \hat{x}$$

$$= B_0 M_0 \pi a^2 l \; \hat{y}$$

$$= \left(\tfrac{1}{5} T\right)\left(10^5 \tfrac{A}{m}\right)(\pi)(0.01m)^2(0.001m)\; \hat{y}$$

$$= \frac{\pi}{5} \times 10^{-2} \hat{y} \quad = 0.00628 \text{ Nm} \, \hat{y}$$