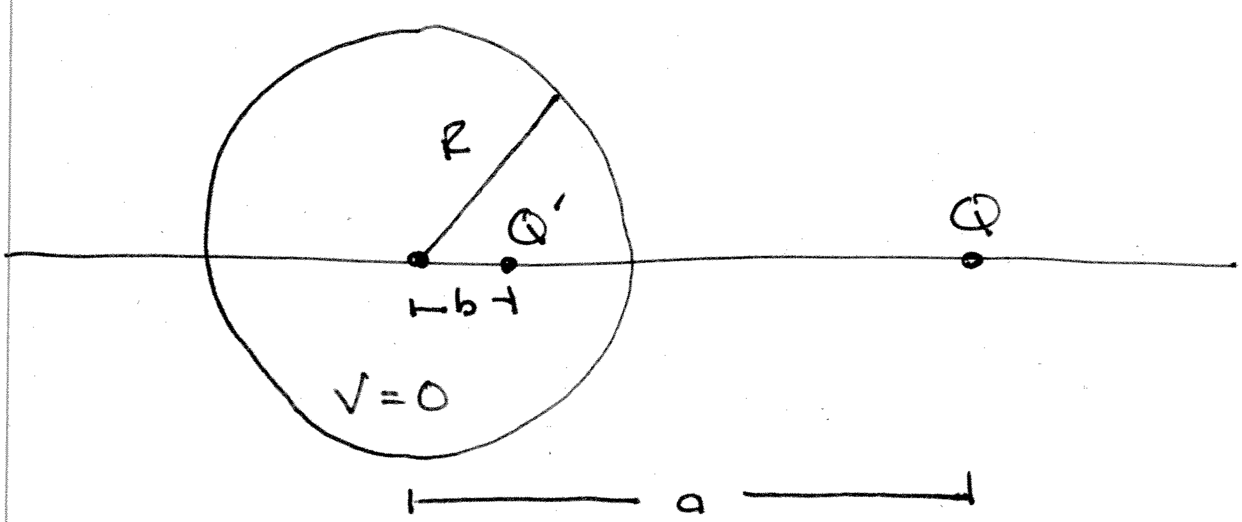
Method of Images - Spherical Systems

To make all points on the surface of a sphere of radius R have $V=0$ when the center of the sphere is a distance a from a point charge Q, place an image charge Q' at $b = R^2/a$ from the center. $Q' = -\frac{R}{a} Q$



$$Q' = -\frac{R}{a} Q$$

$$b = \frac{R^2}{a}$$

$\Rightarrow$ By Gauss' Law, the sphere must have net charge Q'.

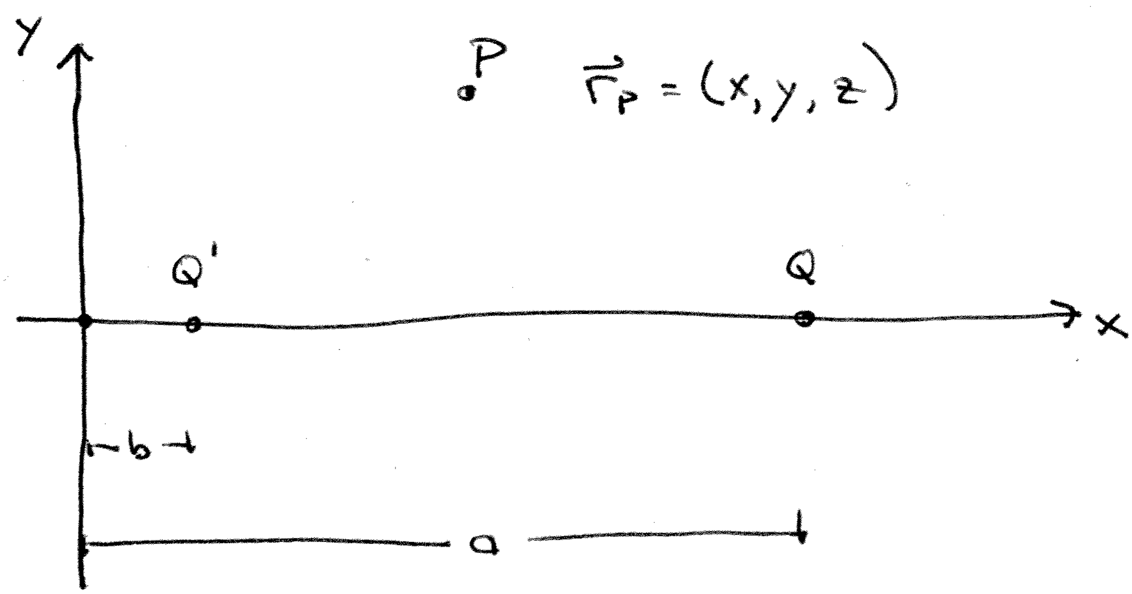
$\Rightarrow$ The potential of a grounded conducting sphere outside of the sphere is

$$V = \frac{kQ'}{r'} + \frac{kQ}{r}$$

r' = distance of image charge to point $\vec{r}$.

$\Rightarrow$ The total charge of the grounded conducting sphere is Q'.

$\Rightarrow$ Using the center of the grounded sphere as the origin



$$V(x,y,z) = \frac{kQ}{\sqrt{(x-a)^2+y^2+z^2}} + \frac{kQ'}{\sqrt{(x-b)^2+y^2+z^2}}$$

Displacement Vecters

From real charge $\vec{r}_{rp} = (x-a, y, z)$

From image charge $\vec{r}_{ip} = (x-b, y, z)$

Field at P

$$\vec{E}_P = \frac{KQ}{r_{rp}^2}\hat{r}_{rp} + \frac{KQ'}{r_{ip}^2}\hat{r}_{ip}$$

Force Q exerts on sphere

$$\vec{F}_{QQ'} = -\frac{KQQ'}{(a-b)^2}\hat{x}$$

Neutral Conducting Sphere

If the sphere is neutral instead of grounded we have to satisfy the following boundary conditions.

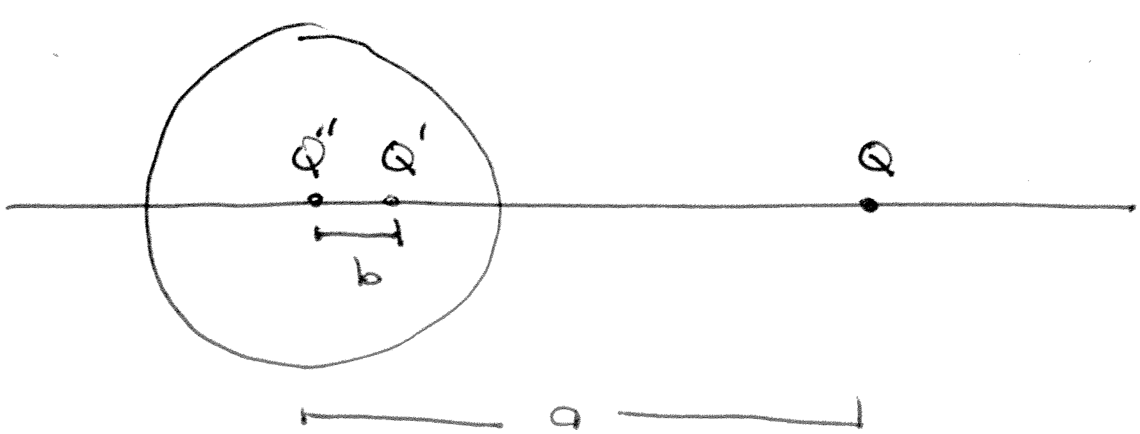
1) $V \to 0$ as $r \to \infty$

2) $\vec{E} \perp \hat{n}$ at $r = R$

3) $Q_{sphere} = 0$

We can satisfy 1) and 2) by using the same image charge Q'

To satisfy 3) while still satisfying 1) and 2) add a uniform surface charge $Q'' = -Q'$ to the surface of the conductor. This additional charge has the same field as a point charge Q'' at the origin.



Naturally, this would also allow us to construct a conducting sphere of an net charge Q_T

$$Q_T = Q'' + Q'$$

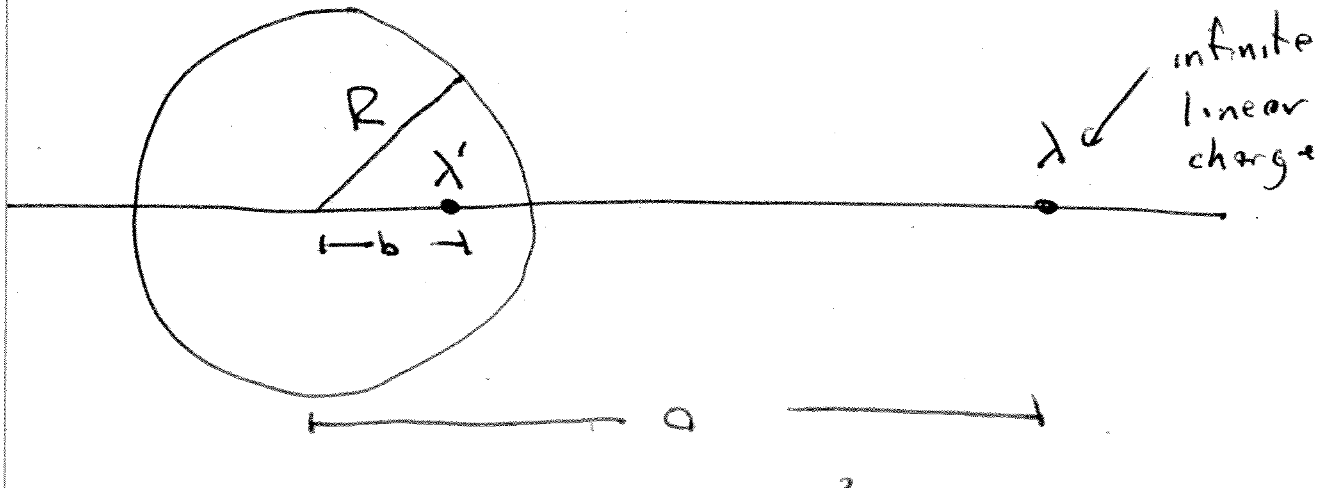
The force the sphere exerts on the external charge is then

$$F = \frac{KQ''Q}{a^2}\hat{x} + \frac{KQ'Q}{(b-a)^2}\hat{x}$$

$\Rightarrow$ The additional charge can also be used to change the potential at the surface by

$$V = \frac{KQ''}{R}$$

$\Rightarrow$ Image Charge for conducting cylinder of radius R

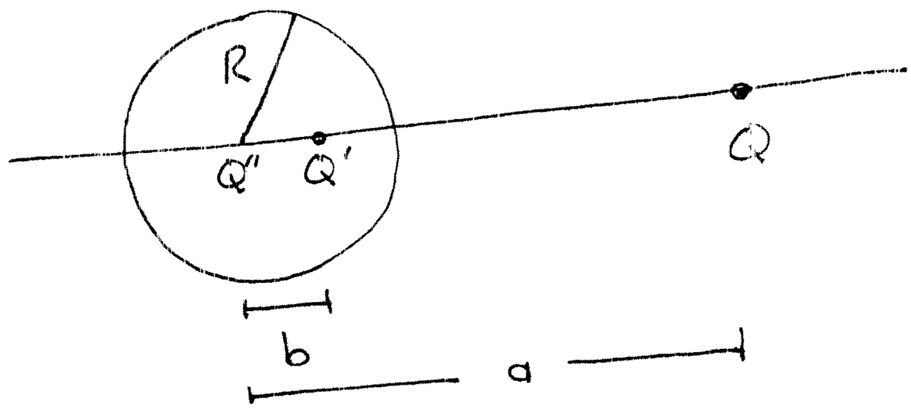


$$\lambda' = \lambda \qquad\qquad b = \frac{R^2}{a}$$

Ex

This allows us to answer a question that has always plagued many of you. What is the actual force exerted by one pith ball on another pith ball?

A charged conducting sphere with charge Q exerts a force F on a point charge Q. Same system as above except $Q_1 = Q$.



$$F = \frac{kQQ'}{(a-b)^2} + \frac{kQQ''}{a^2}$$ (positive if force points to right)

$$b = \frac{R^2}{a} \qquad Q' = -\frac{R}{a}Q$$

$$Q'' = Q - Q' = Q\left(1 + \frac{R}{a}\right)$$

$$F = -KQ^2\left(\frac{R}{a(a-b)^2}\right) + KQ^2\left(\frac{1}{a^2} + \frac{R}{a^3}\right)$$

$$= KQ^2\left(\frac{-Ra^2 + a(a-b)^2 + (a-b)^2R}{a^3(a-b)^2}\right)$$

$$= \frac{KQ^2}{a^2}\left(\frac{a(a-b)^2 + b^2R - 2abR}{a(a-b)^2}\right)$$

But $b = R^2/a$

$$F = \frac{KQ^2}{a^2}\left(\frac{a(a-b)^2 + R^5/a^2 - 2R^3}{a(a-b)^2}\right)$$

$$(a-b)^2 = \left(a - \frac{R^2}{a}\right)^2 = a^2 - 2R^2 + \frac{R^4}{a^2}$$

$$F = \frac{KQ^2}{a^2}\left(\frac{a^3 - 2aR^2 + R^4/a + R^5/a^2 - 2R^3}{a^3 - 2aR^2 + R^4/a}\right)$$

Sometime it doesn't get pretty.

If both spheres are approximated as point charges the force is kQ^2/a^2. So the correction is

$$\frac{F_{sphere}}{F_{point}} = \frac{a(a-b)^2 + b^2R - 2abR}{a(a-b)^2}$$

For our pith balls, $R \sim 1/4$ cm $\quad a \sim 2$ cm

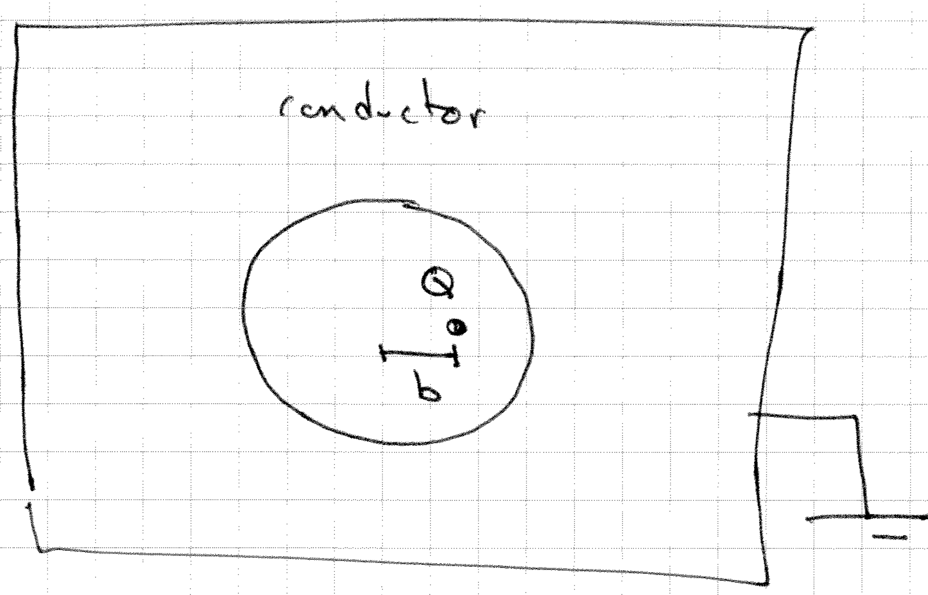
$$b = R^2/a = \frac{\frac{1}{16}}{2} \text{ cm} = \frac{1}{32} \text{ cm}$$

$$\frac{F_{sphere}}{F_{point}} = \frac{(2)\left(\frac{63}{32}\right)^2 + \left(\frac{1}{32}\right)^2 \cdot \frac{1}{4} - 2(2)\left(\frac{1}{32}\right)\left(\frac{1}{4}\right)}{2\left(\frac{63}{32}\right)^2}$$

$$= \frac{2(63)^2 + 1/4 - 32}{2(63)^2} = 0.996$$

So the point charge approximation is very good.

Note, the image charge works both ways. If we want the field of a point charge in a grounded cavity

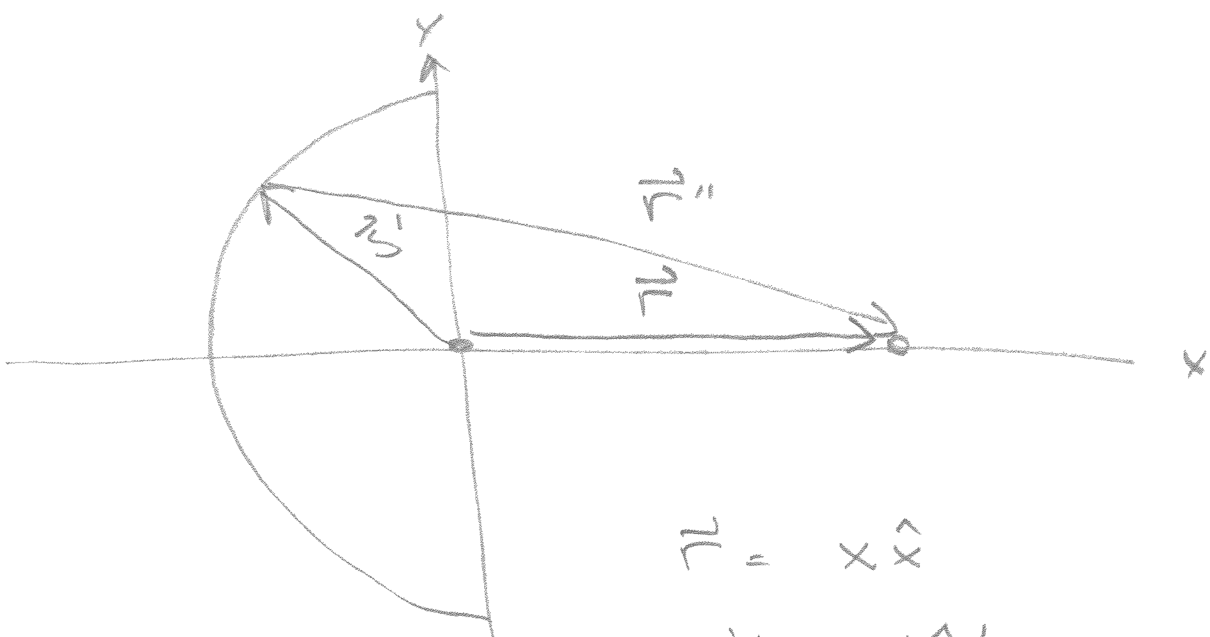


we can use an image charge outside the cavity.

Final Notes about test.

For those of you who did poorly, there are 3 more tests. I will look at a consistent good performance on the final three test as evidence the first test was a fluke.

Problem 4 Started like this



$$\vec{r} = x\hat{x}$$

$$\vec{r}' = s'\hat{s}'$$

$$\vec{r}'' = x\hat{x} - s'\hat{s}'$$

Then I wrote

$$r'' = \sqrt{x^2 + s'^2} = \sqrt{x^2+a^2}$$

$$= \text{constant}$$

Which is obviously wrong

$$r'' = \sqrt{\vec{r}'' \cdot \vec{r}''} = \sqrt{x^2 \hat{x}\cdot\hat{x} + s'^2 \hat{s}'\cdot\hat{s}' - 2xs'\hat{x}\cdot\hat{s}'}$$

$$\hat{x}\cdot\hat{s}' = \hat{x}\cdot(\cos\phi\,\hat{x} + \sin\phi\,\hat{y})$$

$$= \cos\phi$$

$$r'' = \sqrt{x^2 + s'^2 - 2xs'\cos\phi}$$