Integrals

We will build up the fields of complicated objects by integrating over sourse. Many physical laws are best expressed as integrals.

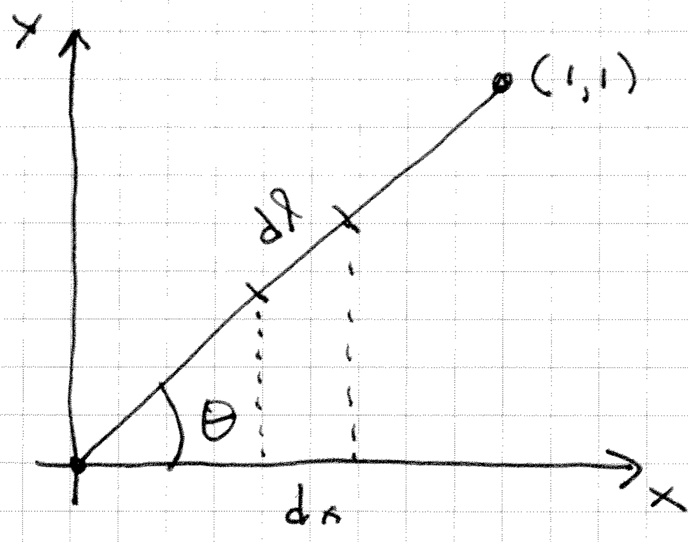
Line Integrals $\displaystyle\int_{\text{curve}} f \, d\ell$

Surface Integrals $\displaystyle\int_{\text{surface}} f \, da$

Volume Integrals $\displaystyle\int_{\text{volume}} f \, d\tau$

To evaluate any integral, imagine dividing the object into small pieces.

Ex Compute $\int d\ell$ along the line from $(0,0)$ to $(1,1)$.



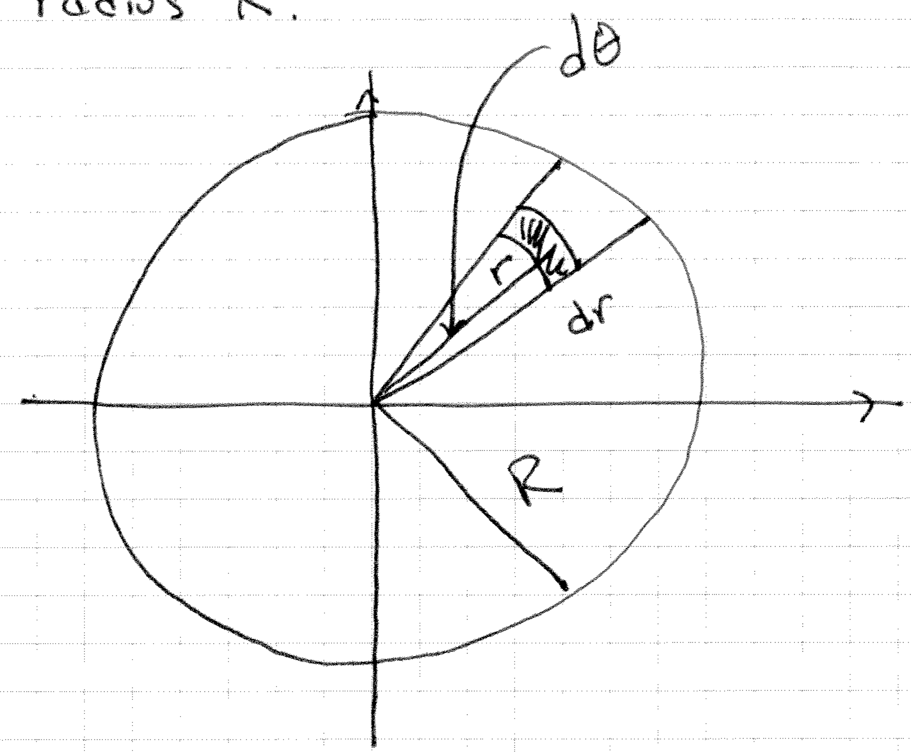
$\Rightarrow$ At the end of the day, we better get the length of the line $\sqrt{2}$.

Sln We don't usually integrate over $d\ell$ directly, but instead integrate over one of the coordinate axes, dx in this case. So we have to write $d\ell$ in terms of dx

$$d\ell = \frac{dx}{\cos\theta} = \sqrt{2}\, dx$$

$$\int_{\text{line}} d\ell = \int_0^1 \sqrt{2}\, dx = \sqrt{2} \checkmark$$

Ex Compute the area of a circle of radius R.



Divide the circle into small pieces as shown above.

The area of each piece is $da = (dr)(r\,d\theta)$

The area of the circle is

$$\text{Area} = \int_{\text{circle}} da = \int_0^R dr \int_0^{2\pi} r\,d\theta$$

$$= \left[\int_0^R r\,dr\right]\left[\int_0^{2\pi} d\theta\right]$$

$$= \frac{R^2}{2} \cdot 2\pi = \pi R^2$$

Integral Theorems

Gradient The total derivative of a function can be written:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

The integral of df between two points is

$$\int_{\vec{r}_a}^{\vec{r}_b} df = f(\vec{r}_b) - f(\vec{r}_a)$$

In Cartesian coordinates, the path element $d\vec{l}$ is

$$d\vec{l} = \hat{x}\, dx + \hat{y}\, dy + \hat{z}\, dz$$

so

$$df = (\nabla f) \cdot d\vec{l}$$

The line integral I of ∇f along a line with $d\vec{l}$ as its path element is

$$I = \int_{\vec{r}_a \to \vec{r}_b} \nabla f \cdot d\vec{l} = \int_{\vec{r}_a}^{\vec{r}_b} df = f(\vec{r}_b) - f(\vec{r}_a)$$

- I is independent of the path taken from $\vec{r}_a$ to $\vec{r}_b$.
- If the path is closed, $\vec{r}_a = \vec{r}_b$, then $I = 0$.

Divergence Thm

$$\int_{\text{volume}} (\nabla \cdot \vec{A})\, d\tau = \oint_{\text{surface}} \vec{A} \cdot d\vec{a}$$

where the surface is closed and encloses the volume.

- $d\vec{a} = \hat{n}\, da$ where $\hat{n}$ is the outward surface normal of the surface.

Dfn Flux The flux or flow of the field $\vec{A}$ out of the closed surface S is defined as

$$\Phi = \oint_S \vec{A} \cdot d\vec{a}$$

$\Rightarrow$ Note we used ϕ for flux in UPII.

- The divergence thm states that the flow of $\vec{A}$ out of the surface is equal to the integral of $\nabla \cdot \vec{A}$ is the surface.

- To make sense $\nabla \cdot \vec{A}$ must act as the source of the flow.

Stoke's Thm

The integral of the field $\vec{A}$ around the closed curve C is equal to the total curl of the field over any surface S bounded by C.

$$\oint_c \vec{A} \cdot d\vec{l} = \int_S (\nabla \times \vec{A}) \cdot d\vec{a}$$

- Chose $\hat{n}$ for S by curling fingers in the direction of C, your thumb points in the direction of $\hat{n}$.

- Stoke's thm states that the total rotation of $\vec{A}$, $\oint_c \vec{A} \cdot d\vec{l}$, is the sum of the individual rotations $\nabla \times \vec{A}$.

Other Integral Formulas

$$\int_V \nabla f \, d\tau = \oint_S f \, d\vec{a}$$

$$\int_S \nabla f \times d\vec{a} = -\oint_C f \, d\vec{l}$$

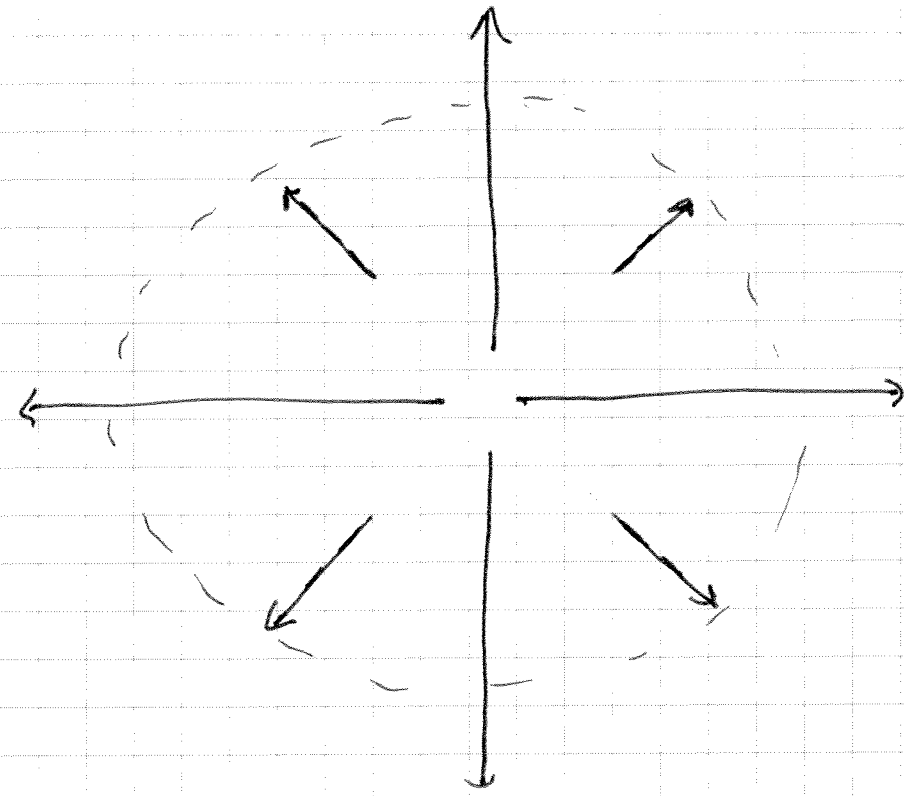
$$\int_V (\nabla \times \vec{A}) \, d\tau = -\oint_S \vec{A} \times d\vec{a}$$

$$\int_S (d\vec{a} \times \nabla) \times \vec{A} = -\oint \vec{A} \times d\vec{l}$$

$$\int_V (f \nabla^2 g + \nabla f \cdot \nabla g) \, d\tau = \oint_S (f \nabla g) \cdot d\vec{a}$$

$$\int_V (f \nabla^2 g - g \nabla^2 f) \, d\tau = \oint_S (f \nabla g - g \nabla f) \cdot d\vec{a}$$

Ex Consider the field $\vec{A} = \gamma \vec{r}$ where γ is a constant. The field points outward from the origin and grows linearly with $\vec{r}$.



The flux Φ out of a sphere of radius R is

$$\Phi = \int_{\text{surface}} \vec{A} \cdot d\vec{a}$$

$$\cancel{d\vec{a} = \hat{r}\,(r \sin\theta \, d\theta)}$$

normal

$$d\vec{a} = \underbrace{(r \sin\theta \, d\phi)(r\, d\theta)}_{\text{area of small square of surface}} \, \hat{r}$$

$$\Phi = \oint_S (\gamma \vec{r}) \cdot \hat{r} \, (R \sin\theta \, d\phi)(R \, d\theta)$$

$$\vec{r} = r\hat{r} \implies \vec{r} \cdot \hat{r} = r$$

on sphere $R = r$.

$$\Phi = \gamma R^3 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta$$

$$= 2\pi R^3 \gamma \int_0^{\pi} \sin\theta \, d\theta$$

$$= -2\pi R^3 \gamma \cos\theta \Big|_0^{\pi}$$

$$= 4\pi R^3 \gamma$$

$\implies$ Much more simply, since $|\vec{A}|$ is constant on S,

$$\Phi = |\vec{A}| \cdot \text{Area} = \gamma R \cdot 4\pi R^2$$

Now check the divergence thm,

$$\vec{A} = \underbrace{\gamma r}_{A_r}\,\hat{r} + 0\,\hat{\theta} + 0\,\hat{\phi}$$

From front cover, in spherical,

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\gamma r^3$$

$$= \frac{3\gamma r^2}{r^2} = 3\gamma$$

$$\int_{\text{Volume}} \nabla\cdot\vec{A}\,d\tau = \int_{\text{Volume}} 3\gamma\,d\tau$$

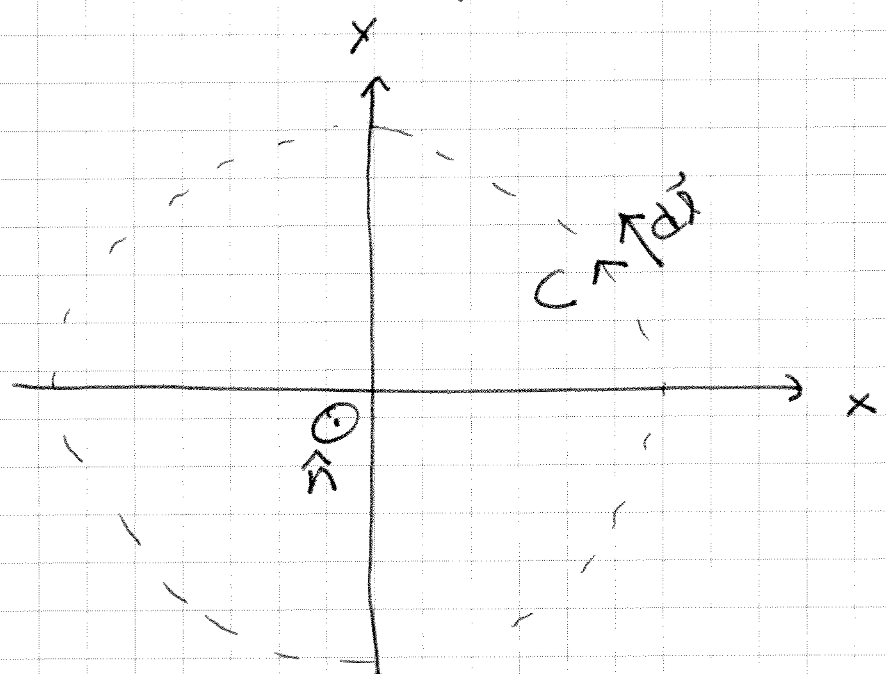
$$= 3\gamma\int_{\text{Volume}} d\tau = 3\gamma\cdot\frac{4}{3}\pi R^3$$

$$= 4\gamma\pi R^3 \checkmark$$

Ex Pick a field that is purely rotational

$$\vec{A} = \gamma \hat{\phi}$$

and apply Stoke's Thm for a circular curve C with path element $d\vec{l} = R\,d\phi\,\hat{\phi}$



$\hat{n} = \hat{z}$ by RHR

$$\oint_c \vec{A} \cdot d\vec{l} = \oint_c \vec{A} \cdot (R\,d\phi\,\hat{\phi})$$

$$= \oint_c (\gamma\hat{\phi}) \cdot (R\hat{\phi}\,d\phi) = R\gamma \oint_c d\phi$$

$$= R\gamma \int_0^{2\pi} d\phi = 2\pi R\gamma$$

Compute Curl (cylindrical)

$$A_s = 0, \quad A_\phi = \gamma, \quad A_z = 0$$

$$\nabla \times \vec{A} = \underbrace{-\left(\frac{\partial A_\phi}{\partial s}\right)}_{=\,0}\hat{s} + \underbrace{\frac{1}{s}\left(\frac{\partial}{\partial s}(s A_\phi)\right)\hat{z}}_{=\,\left(\frac{1}{s}\frac{\partial}{\partial s}\gamma s\right)\hat{z}}$$

$$= \frac{\gamma}{s}\hat{z}$$

<u>Now Compute</u> $\quad d\vec{a} = \underbrace{s\,d\phi}\,\underbrace{ds}\,\hat{z}$

$$\int_S (\nabla \times \vec{A}) \cdot d\vec{a} = \int_0^{2\pi} d\phi \int_0^R s \cdot \frac{\gamma}{s}\,\hat{z}\cdot\hat{z}$$

$$= \gamma \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^R ds}_{R}$$

$$= 2\pi R \gamma \checkmark$$