Laplace's Equ - Cylindrical Coordinates

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Trivial Solutions $1, \ln(s), \phi, z$

Axially Radial (B.C. does not depend on ϕ, z)

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0$$

$$V = 1, \ln(s)$$

General Solution

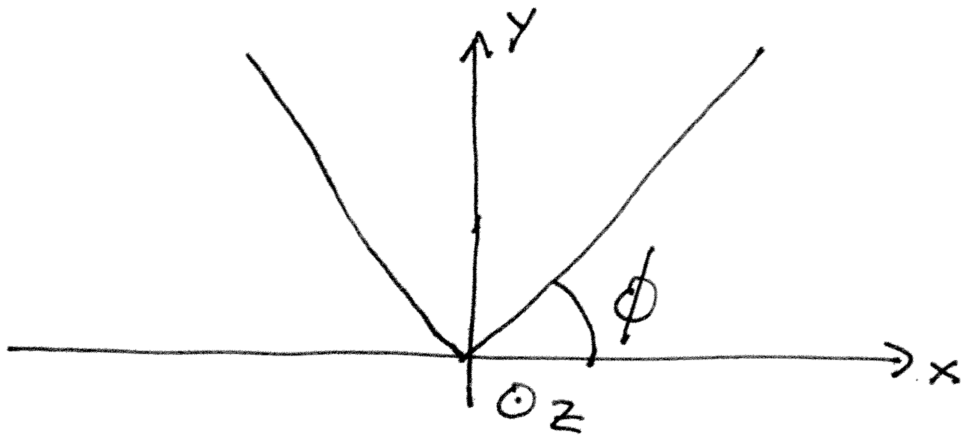
$$V = a_1 + a_2 \ln(s)$$

Azimuthal (B.C. does not depend on s, z)

$$V = V_0 \phi + C \quad \text{general solution}$$

TOPS 35502

<u>Ex</u> Compute electric field in wedge where one face at $V(60°) = 0$ and the other face at $V(120°) = V_0$



<u>Sln</u> Boundary conditions depend only on ϕ

<u>General Solution</u>

$$V(\phi) = a_1 + a_2\phi$$

<u>Boundary Conditions</u>

$$V(60°) = V\left(\frac{\pi}{3}\right) = a_1 + a_2 \cdot \frac{\pi}{3} = 0$$

$$V(120°) = V\left(\frac{2\pi}{3}\right) = a_1 + a_2 \cdot \frac{2\pi}{3} = V_0$$

$$a_2 \cdot \frac{\pi}{3} = V_0 \implies a_2 = \frac{3V_0}{\pi}$$

$$a_1 = -\frac{3}{\pi} a_2 = -\frac{3}{\pi} \cdot \frac{3V_0}{\pi} = -V_0$$

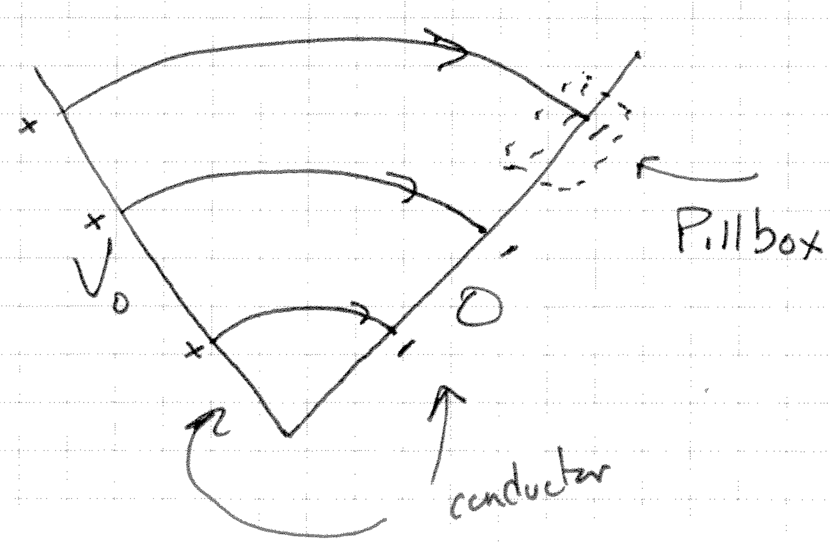
$$V(\phi) = -V_0 + \frac{3V_0}{\pi}\phi$$

<u>Electric Field</u>

$$\vec{E} = -\nabla V = -\frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\phi}$$

$$= -\frac{1}{s}\left(\frac{3V_0}{\pi}\right)\hat{\phi} = -\frac{3V_0}{\pi s}\hat{\phi}$$

⇒ Note field correctly points in $-\hat{\phi}$ direction from high potential to low potential.



Use Gaussian Pillbox to find charge density

$$\phi = -\left|E\left(\frac{\pi}{3}\right)\right| A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\sigma = -\left|E\left(\frac{\pi}{3}\right)\right| \varepsilon_0$$

$$= -\frac{3 V_0}{\pi s} \varepsilon_0 \implies \text{Charge on } \phi = 60° \text{ conductor}$$

Translation Symmetry (no z dependence)

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Separation $V = S(s)\,\Phi(\phi)$

$$\underbrace{\frac{1}{S(s)}\, s\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)}_{= K^2} + \underbrace{\frac{1}{\Phi(\phi)}\frac{\partial^2 \Phi}{\partial \phi^2}}_{= -K^2} = 0$$

Solve for Φ

$$\frac{d^2\Phi}{d\phi^2} + k^2\Phi = 0$$

Solutions $\sin k\phi, \cos k\phi$

To be continuous at $\phi = 0$, $\phi = 2\pi$, k must be an integer $k = n$.

Solutions $\sin n\phi, \cos n\phi$.

Solve $S(s)$

$$s\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) - n^2 S = 0$$

Try $S(s) = s^a$

$$s\frac{\partial}{\partial s}\left(a s^a\right) - n^2 s^a = 0$$

$$a^2 s^a - n^2 s^a = 0$$

$$n^2 = a^2 \Rightarrow a = \pm n$$

Solutions $\quad 1, \ \ln(s), \ \phi,$

$$s^n \cos n\phi \qquad s^{-n} \cos n\phi$$

$$s^n \sin n\phi \qquad s^{-n} \sin n\phi$$

Orthogonality

$$\int_0^{2\pi} \sin n\phi \, \sin m\phi \, d\phi = \pi \delta_{nm}$$

$$\int_0^{2\pi} \cos n\phi \, \cos m\phi \, d\phi = \begin{cases} \pi \delta_{nm} & \text{if } n,m>0 \\ 2\pi & \text{if } n=m=0 \end{cases}$$

Ex An infinite cylinder of radius a has a potential $V(\phi) = V_0 \cos^2\phi$ established on its surface. Compute field outside cylinder.

General Solution

$$V(s,\phi) = \sum A_n s^n \cos n\phi + B_n s^n \sin n\phi$$

$$+ C_n s^{-n} \cos n\phi + D_n s^{-n} \sin n\phi$$

Boundary Conditions

$$V(\infty) = 0 \implies A_n = B_n = 0$$

Work on Potential

$$\cos^2\phi = \frac{1}{2} + \frac{1}{2}\cos 2\phi$$

Apply Boundary Condition

$$V(a,\phi) = \frac{V_0}{2} + \frac{V_0}{2}\cos 2\phi$$

$$= \sum_n C_n a^{-n} \cos n\phi + D_n a^{-n} \sin n\phi$$

Using orthogonality, $C_0 = \frac{V_0}{2}$

$$\frac{V_0}{2} \cos 2\phi = C_2 \, a^{-2} \cos 2\phi$$

with all other terms zero.

$$C_2 = \frac{V_0 a^2}{2}$$

$$V(s, \phi) = \frac{V_0}{2} + \frac{V_0 a^2}{2 s^2} \cos 2\phi$$

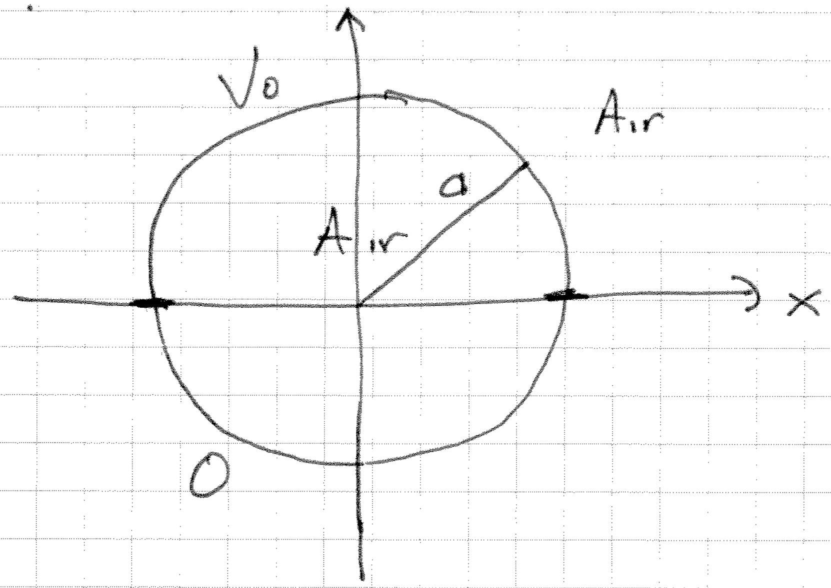
<u>Compute Field</u>

$$\vec{E} = -\nabla V$$

$$= -\frac{\partial V}{\partial s} \hat{s} + \frac{1}{s} \frac{\partial V}{\partial \phi} \hat{\phi}$$

$$= + \frac{V_0 a^2}{s^3} \cos 2\phi \, \hat{s} + \frac{V_0 a^2}{s^3} \sin 2\phi \, \hat{\phi}$$

Ex Infinite cylinder of radius a with top half at V_0 and bottom half at 0.



Compute field inside

Orthogonality

$$\int_0^{2\pi} \cos n\phi \cos m\phi \, d\phi = \begin{cases} \delta_{nm}\pi & \text{if } n,m>0 \\ 2\pi & \text{if } n=m=0 \end{cases}$$

$$\int_0^{2\pi} \sin n\phi \sin m\phi \, d\phi = \delta_{nm}\pi$$

Inside, discard s^{-n} terms because they blow up at origin.

General Solution

$$V(s,\phi) = \sum_n A_n s^n \cos n\phi + B_n s^n \sin n\phi$$

Boundary Condition

$$V(a,\phi) = \begin{cases} 0 < \phi < \pi & V = V_0 \\ \pi < \phi < 2\pi & V = 0 \end{cases}$$

$$= \sum_n A_n a^n \cos n\phi + B_n a^n \sin n\phi$$

Fourier's Trick

$$\int_0^{2\pi} V(a,\phi) \cos m\phi = \sum_n A_n a^n \delta_{nm} \pi$$

$n, m > 0$

$$\int_0^{\pi} V_0 \cos m\phi \, d\phi = A_m a^m \pi$$

$$\int_0^{\pi} V_0 \cos m\phi \, d\phi = \frac{V_0}{m} \sin m\phi \Big|_0^{\pi}$$

$$= 0$$

$$\Rightarrow A_m = 0 \qquad m > 0$$

Likewise,

$$B_n a^m \pi = \int_0^{\pi} V_0 \sin m\phi \, d\phi = \frac{-V_0}{m} \cos m\phi \Big|_0^{\pi}$$

$$= \begin{cases} 0 & m \text{ even} \\ \frac{2V_0}{m} & m \text{ odd} \end{cases}$$

$$B_m = \frac{2V_0}{\pi m a^m} \qquad m \text{ odd}$$

$$= 0 \qquad m \text{ even}$$

Finally, the $n=0$, $m=0$ cosine case

If $m=n=0$, $\int_0^{2\pi} \cos n\phi \cos m\phi \, d\phi = \int_0^{2\pi} d\phi \, \delta_{nm}$

$$= 2\pi \delta_{nm}$$

So

$$2\pi A_0 a^0 = \int_0^{2\pi} \cos(0)\, V(a,\phi)\, d\phi$$

$$= \int_0^{\pi} V_0\, d\phi = \pi V_0$$

$$A_0 = \frac{V_0}{2}$$

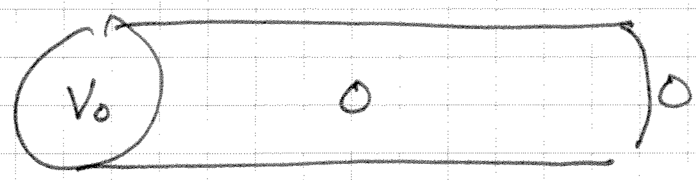
Full Solution

$$V(s,\phi) = \frac{V_0}{2} + \sum_{n\,\text{odd}} \frac{2V_0}{n\pi a^n} s^n \sin n\phi$$

Full Solution Cylindrical

B.C. depends on s, ϕ, z

such as



General Solution

$$\left[J_\nu(sk) + N_\nu(sk) \right] \times \left[\sin \nu\phi + \cos \nu\phi \right] \times \left[e^{kz} + e^{-kz} \right]$$

J_ν Bessel Function

N_ν Neuman Function