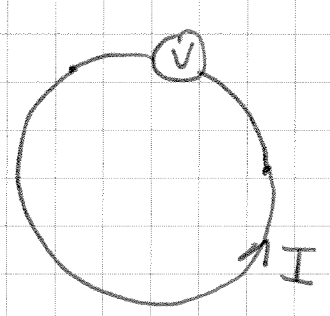
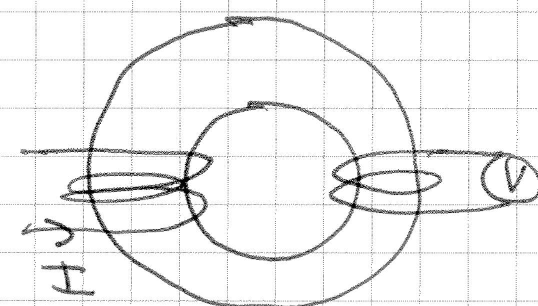
Magnetic Energy

Written at Quidditch World Cup,
Orlando Fl.

If a current $I(t)$ flows through a circuit with self-inductance or mutual inductance, a voltage (emf) will be measured across the circuit.





By Faraday's Law, the emf which measures as a voltage is given by

$$\text{emf} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt} LI = -L\frac{dI}{dt}$$

$$\text{emf} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt} M_{ab} I = -M_{ab}\frac{dI}{dt}$$

This allows the calculation of the work required to establish the current.

$$\text{Work} = W = \int \underset{\text{Power}}{P}\, dt = \int I \Delta V\, dt$$

$$W = \int I \left(L \frac{dI}{dt} \right) dt$$

$$= L \int I\, dI = \frac{1}{2} L I^2$$

$=$ Energy stored in the magnetic fields of the current.

Energy Stored in an Inductor

$$U = \frac{1}{2} L I^2$$

This energy can also be written in terms of the vector potential, recall

$$\Phi_m = \int_C \vec{A} \cdot d\vec{l}$$

$$\Phi_m = L I$$

$$U = \frac{1}{2} L I^2 = \frac{1}{2} \Phi_m I$$

$$= \frac{1}{2} I \oint_c \vec{A} \cdot d\vec{l}$$

$$= \frac{1}{2} \int_c (\vec{A} \cdot \vec{I})\, dl$$

In terms of the current density $\vec{J}$,

$$U = \frac{1}{2} \int_V (\vec{A} \cdot \vec{J})\, d\tau$$

Compare this with the energy density in the electric field.

$$U = \frac{1}{2} \int_V \rho V \, d\tau$$

We also wrote the energy density in terms of the electric field as $u = \frac{1}{2} \varepsilon_0 \vec{E}^2$

Let's find the corresponding magnetic version.

Consider an infinitely long solenoid carrying current I. The energy stored in a length l of the solenoid is.

$$U = \frac{1}{2} L I^2 = \frac{1}{2}\left(\mu_0 n^2 A l\right) I^2$$

$$= \frac{1}{2\mu_0}\left(\mu_0 n I\right)^2 A l$$

$$= \frac{1}{2}\frac{B^2}{\mu_0} A l$$

The volume of the solenoid is Al, so the energy per unit volume is

$$u = \frac{U}{V} = \frac{U}{Al} = \frac{B^2}{2\mu_0}$$

$\Rightarrow$ Since magnetic field is magnetic field, this is the energy density of the magnetic field.

Now suppose we fill the solenoid with a linear magnetic material μ_r. The resulting magnetic field is

$$B = \mu_r B_0$$

The new inductance is

$$L = \frac{\Phi_m}{I} = \frac{NBA}{I} = \frac{N(\mu_r \mu_0 n I)A}{I}$$

$$= \mu_0 \mu_r n^2 A l$$

$$n = \frac{N}{l}$$

$\Rightarrow$ The inductance is increased by μ_r

The energy at current I is the

$$U = \frac{1}{2} L I^2 = \frac{1}{2}(\mu_0 \mu_r n^2 A l) I^2$$

$$= \frac{1}{2}(\mu_0 \mu_r n I)(n I) A l$$

Recall,

$$\mu_0 \mu_r \vec{H} = \vec{B}$$

$$\Rightarrow \quad H = n I$$

which makes sense because H only involves the free currents.

$$\Rightarrow \quad U = \frac{1}{2} \vec{B} \cdot \vec{H} \; A l$$

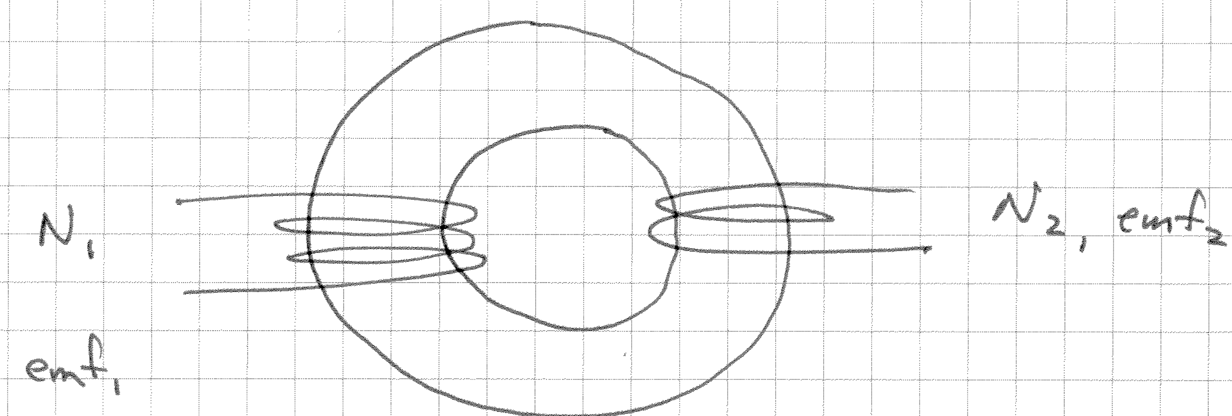
⇒ The magnetic energy density in a magnetic material is

$$u = \frac{1}{2} \vec{B} \cdot \vec{H}$$

⇒ Compare with the electric energy density is a dielectric material.

$$u = \frac{1}{2} \vec{D} \cdot \vec{E}$$

<u>Transformer</u> - Consider a device wound with two coils on a iron core



The magnetic field is uniform in the iron ring, so the magnetic flux through each loop of either circuit is constant. Let the flux through one loop be Φ_0.

The flux through loop 1 is

$$\Phi_{m,1} = N_1 \Phi_0$$

and loop 2

$$\Phi_{m,2} = N_2 \Phi_0$$

The emf around loop 1 is

$$emf_1 = -N_1 \frac{d\Phi_0}{dt}$$

and loop 2

$$emf_2 = -N_2 \frac{d\Phi_0}{dt}$$

The ratio of the emfs is

$$\frac{emf_2}{emf_1} = \frac{-N_2 \frac{d\Phi_0}{dt}}{-N_1 \frac{d\Phi_0}{dt}} = \frac{N_2}{N_1}$$

$\Rightarrow$ Transformers can be used to step up (increase) or step down (decrease) voltage.

⇒ Commercial transformers are fairly efficient so

Power In $\approx$ Power Out

⇒ Each time a transformer goes through an AC cycle it is driven around its hysteresis loop and heated by the area within the loop $\frac{1}{2}\vec{B}\cdot\vec{H}$ multiplied by its volume.

⇒ Real transformers work to prevent this heating by using laminar ferrous materials.