Magnetic Force

The force exerted by a magnetic field on a moving charged particle q is

$$\vec{F} = q\vec{v} \times \vec{B} \qquad \text{(Lorentz Force)}$$

where $\vec{v}$ is the velocity

If the moving charge is in the form of a current

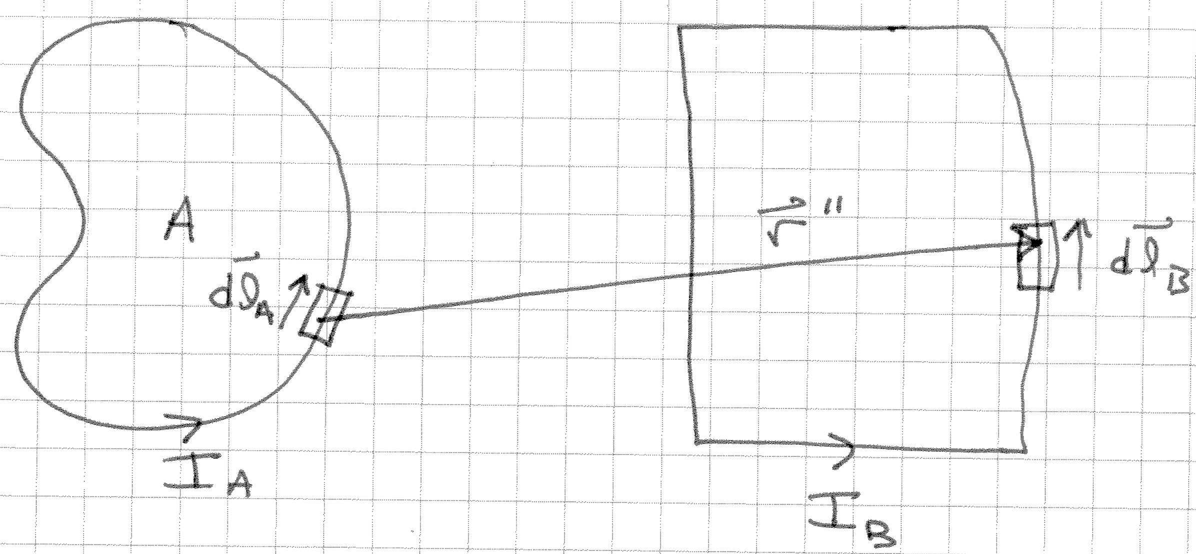
$$\vec{F} = \int_C \vec{I}(\vec{r}) \times \vec{B}(\vec{r})\, dl$$

$$= \int_C I\, d\vec{l} \times \vec{B}(\vec{r})$$

$$= \int_S \vec{K}(\vec{r}) \times \vec{B}(\vec{r})\, da$$

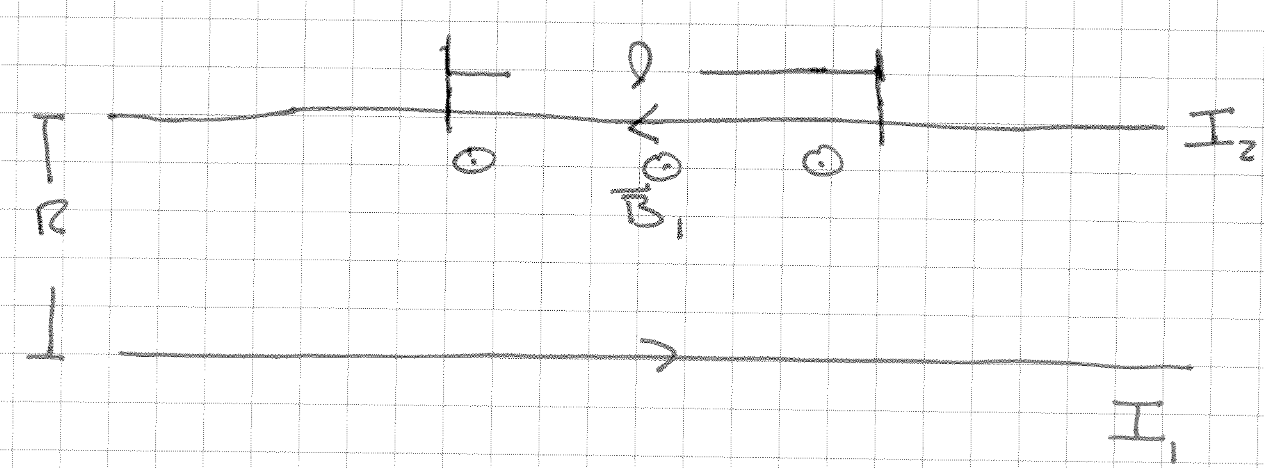
$$= \int_V \vec{J}(\vec{r}) \times \vec{B}(\vec{r})\, d\tau$$

The force one closed circuit A exerts on a second closed circuit B is then



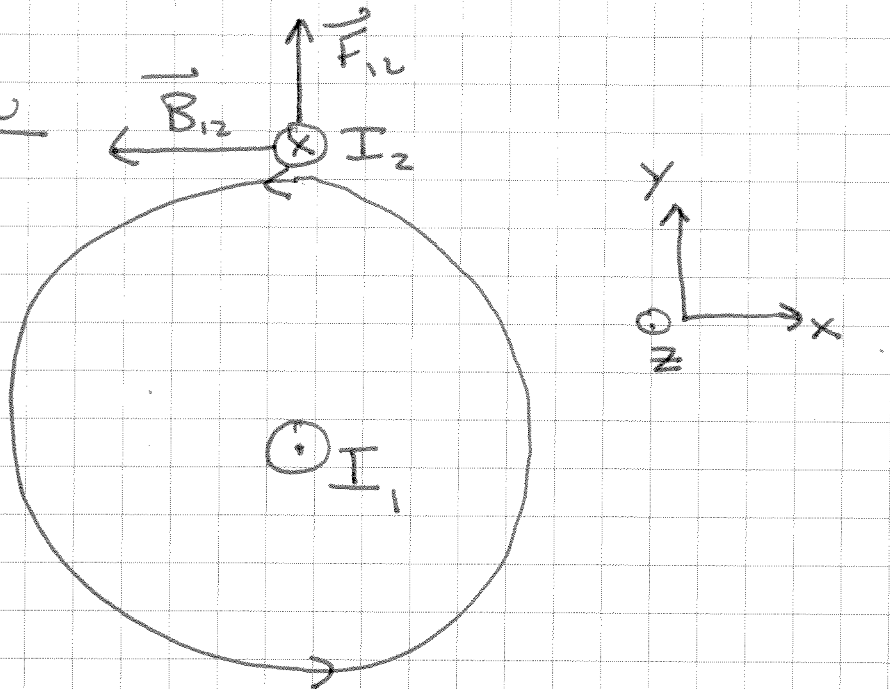
$$\vec{F}_{AB} = \frac{\mu_0 I_A I_B}{4\pi} \oint_{C_A} \oint_{C_B} d\vec{l}_B \times \left(\frac{d\vec{l}_A \times \hat{r}\,''}{(r'')^2} \right)$$

Ex Force per unit length that one long wire exerts on a second parallel wire



Compute force the bottom wire exerts on a length l of the top wire.

End View



The magnetic field of the bottom wire at the top wire

$$B_{12} = \frac{\mu_0 I_1}{2\pi R}$$

The magnetic force

$$|\vec{F}_{12}| = I_2 \int_0^l |d\vec{l}_2 \times \vec{B}_{12}|$$

$$\vec{B}_{12} = -B_{12}\hat{x} \qquad d\vec{l}_2 = -dl\,\hat{z}$$

$$d\vec{l}_2 \times \vec{B}_{12} = dl\,\hat{z} \times B_{12}\,\hat{x}$$
$$= dl\,B_{12}\,\hat{y}$$

$$|F_{12}| = I_2\,B_{12} \int_0^l dl$$

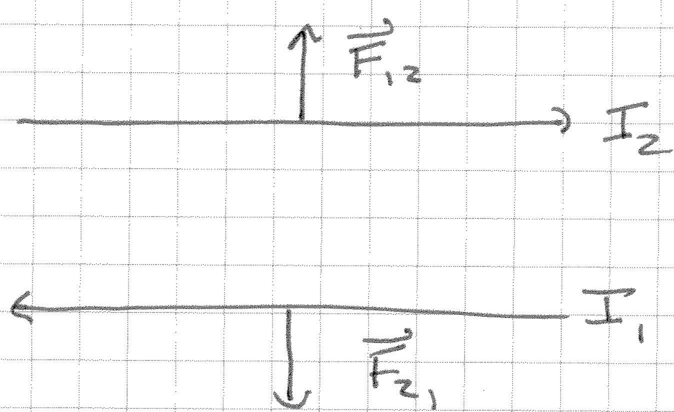
$$= I_2\,B_{12}\,l$$

$$\frac{|F_{12}|}{l} = \frac{\mu_0 I_1 I_2}{2\pi R}$$

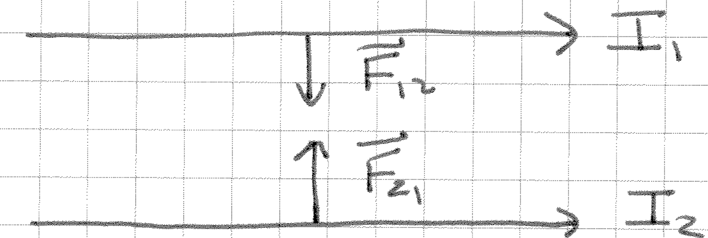
For example, house wiring is fused at 20 A and the conductors are about 1 cm apart.

$$\frac{|F_{12}|}{l} = \frac{(4\pi\times10^{-7}\,Tm/A)(20\,A)^2}{2\pi\,(0.01\,m)} = 8\times10^{-3}\,\frac{N}{m}$$

Opposite Currents Repel

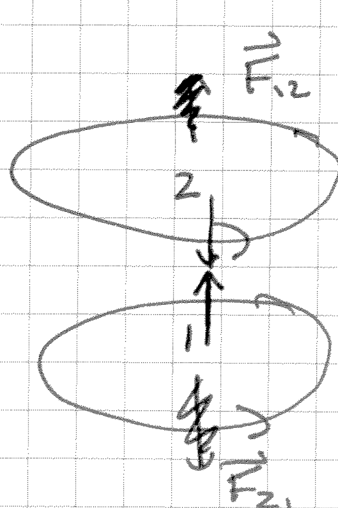


Like Currents Attract

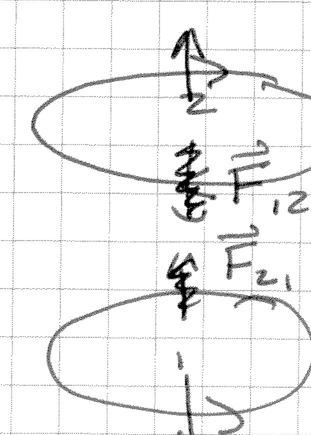


If we wrap these currents in loops

Same Direction Attractive



Opposite Orientation Repulsive



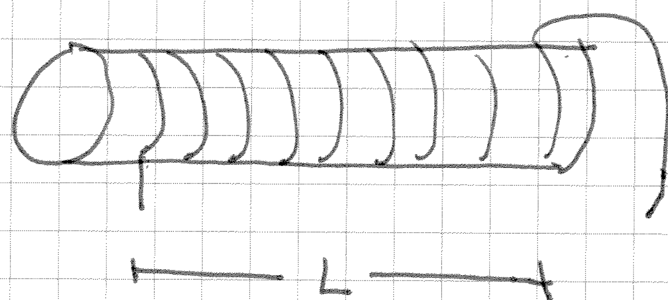
If the two loops are close together,

$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi R}$$

is a good approximation for the force.

Ex

Dfn Solenoid - A set of wraps of wire on a tube is called a solenoid.



N = number of wraps or "turns"

$$n = \frac{N}{L} = \text{turns per unit length}$$

If the solenoid is long ("infinite") the magnetic field outside is zero and the magnetic field inside is

$$B_{sol} = \mu_0 n I$$

(which we will show using Ampere's Law)

Ex My favorite solenoid is wound on a plastic sewer pipe with $N = 79$ turns over $L = 79\,cm$ for a turn spacing of $1\,cm$

The force one turn exerts on another is about

$$\frac{F}{\ell} = \frac{\mu_0 I_1 I_2}{2\pi R}$$

$$\frac{F}{\ell} = \cancel{\mu_0 I_1 I} = \frac{\mu_0 I^2}{2\pi R} \quad \text{at } 20\,A$$

$$= \frac{(4\pi \times 10^{-7}\,T\,m/A)(20\,A)^2}{2\pi\,(0.01\,m)} = 8\times10^{-3}\,N/m$$

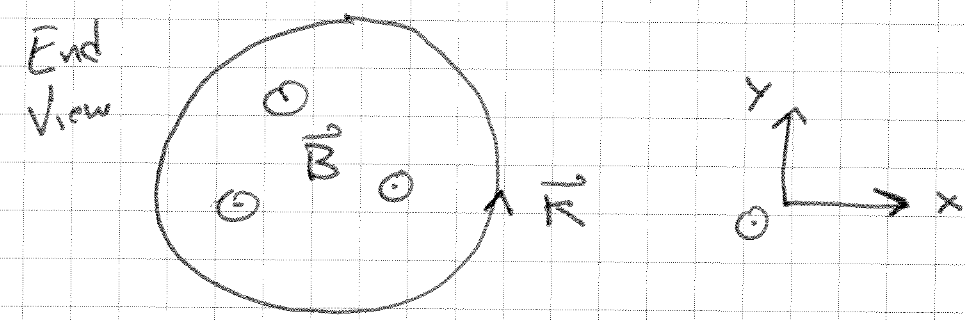
The length of one turn is $2\pi r$, $r \sim 5\,cm$.

$$F = \frac{F}{\ell} \cdot 2\pi r$$

$$= (8\times10^{-3}\,N/m)(2\pi)(0.05\,m) =$$

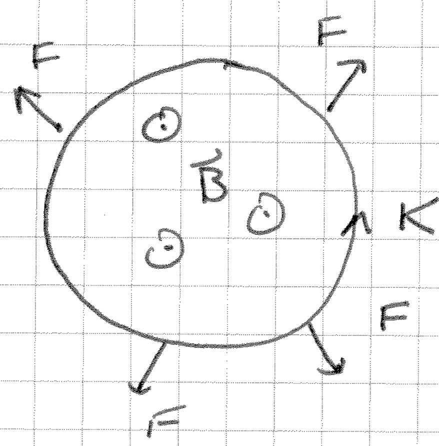
$$= 8\pi\times10^{-4}\,N \quad (\text{attractive})$$

If the solenoid is tightly wound, we can approximate the current as a surface current $\vec{K}$



$$\vec{K} = nI\hat{\phi} \qquad B_{sol} = \mu_0 K$$

Ex Pressure on Long Solenoid



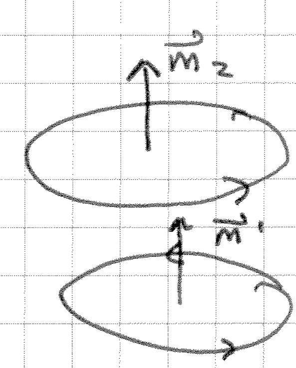
<u>Magnetic Pressure</u>

$$P = \left| \vec{K} \times \frac{1}{2}\left(\vec{B}_{inside} + \vec{B}_{outside}\right)\right|$$

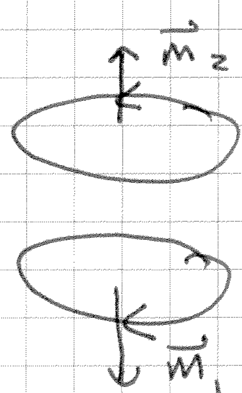
$$= \cancel{\tfrac{1}{1}}\, K \cdot \frac{1}{2}\left(\mu_0 K + 0\right)$$

$$= \frac{1}{2}\mu_0 K^2 = \frac{1}{2}\mu_0 n^2 I^2 \text{ outward}$$

Eventually, we will find current loops have magnetic moments, $\vec{m}$.

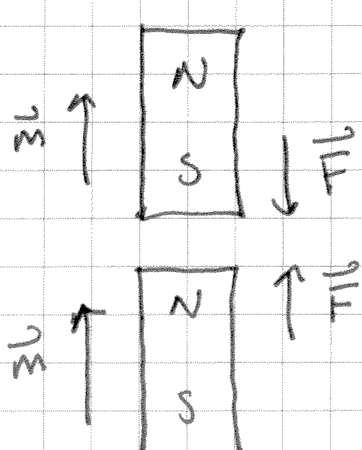


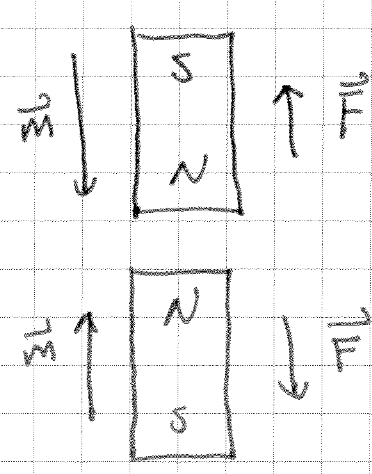
Aligned moments attract.



Opposite moments repel

Label the tail of the moment vector South (S) and the head North (N).



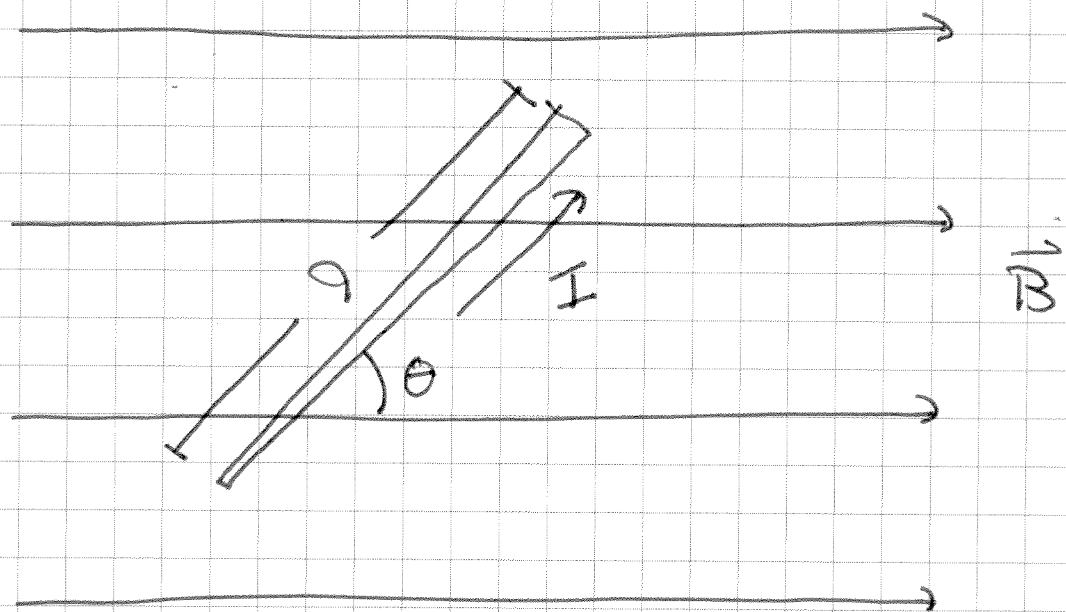


Opposite poles attract, Like poles repel

All magnetic field is made by moving charge.

$\Rightarrow$ There must be the equivalent of a set of current loops in a magnetic material.

<u>Ex</u> Force on straight wire segment at angle θ to uniform magnetic field B. Length of segment ℓ.



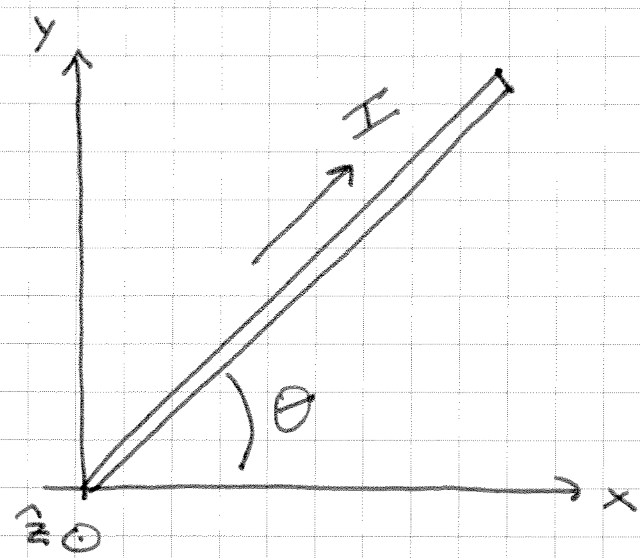
$$d\vec{F} = I\, d\vec{\ell} \times \vec{B}$$

$\Rightarrow$ Force is into page by RHR.

$$|d\vec{F}| = |I\, d\vec{\ell} \times \vec{B}| = I\, d\ell\, B \sin\theta$$

$$|\vec{F}| = \left| \int_{\text{wire}} d\vec{F} \right| = I B \sin\theta \int_{\text{wire}} d\ell = I B \ell \sin\theta$$

Lets solve this by doing the path integral for real!



The wire's location is given by

$$y' = \tan\theta \, x'$$

$$dy' = \tan\theta \, dx'$$

$$d\vec{l} = dx'\,\hat{x} + dy'\,\hat{y}$$

$$= dx'\,\hat{x} + \tan\theta \, dx'\,\hat{y}$$

$$\vec{B} = B_0\,\hat{x}$$

$$d\vec{l}' \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & dx'\tan\theta & 0 \\ B_0 & 0 & 0 \end{vmatrix} = -B_0 \tan\theta \, dx'\,\hat{z}$$

$$\vec{F} = \int_{wire} d\vec{F} = I\int_0^{l\cos\theta} -B_0 \tan\theta \, dx' \, \hat{z}$$

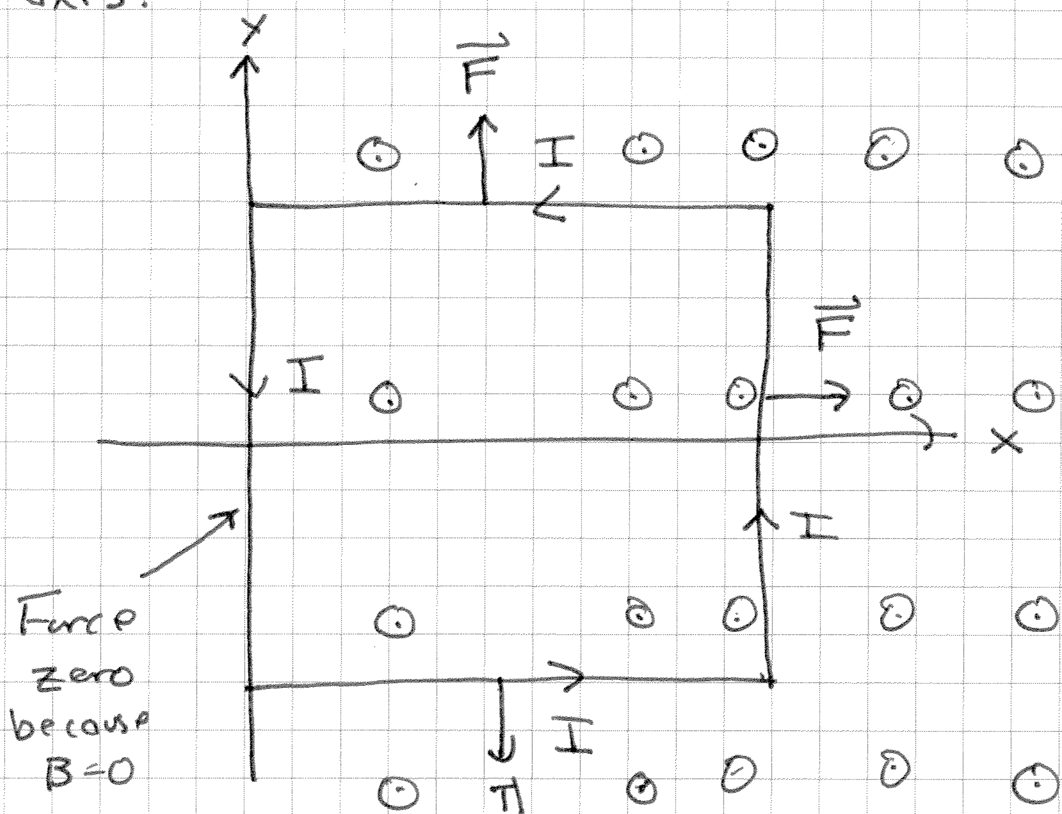
$$x' \in [0, l\cos\theta]$$

$$\vec{F} = -IB_0 \tan\theta \, \hat{z} \int_0^{l\cos\theta} dx'$$

$$= -IB_0 \, l \tan\theta \cos\theta \, \hat{z}$$

$$= -B_0 l I \sin\theta \, \hat{z} \quad \checkmark$$

Ex A square loop of wire with sides a is in a non-uniform magnetic field $\vec{B} = \frac{B_0 x}{a} \hat{z}$
The loop lies in the x-y plane with one edge along the y-axis centered on the x-axis. The wire carries current I CCW.



B points out of the page and gets stronger as x increases.

First, is this a possible magnetostatic field. If it is what current produced the field?

Test $\nabla \cdot \vec{B} = 0$

$$\nabla \cdot \vec{B} = \frac{B_0}{a} \frac{\partial x}{\partial z} = 0 \checkmark$$

What current produced the field?

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{B_0}{a} x \end{vmatrix}$$

$$= -\frac{B_0}{a} \hat{y} = \mu_0 \vec{J}$$

$$\vec{J} = -\frac{B_0}{\mu_0 a} \hat{y} \Rightarrow$$ Constant current density in $-y$ direction.

Calculate force on loop

1) Force on left side is zero because $\vec{B} = 0$

2) Force on top and bottom cancel.

3) Force on right side

$$\vec{B} = \frac{B_0}{a} \cdot a\,\hat{z} = B_0\,\hat{z}$$

$$d\vec{l}' = \hat{y}\,dy'$$

$$d\vec{l}' \times \vec{B} = B_0\,dy'\,(\hat{y} \times \hat{z}) = B_0\,dy'\,\hat{x}$$

$$\vec{F} = \int_{-a/2}^{a/2} I\,B_0\,dy'\,\hat{x}$$

$$= I\,B_0\,a\,\hat{x}$$