Magnetic Multipole Expansion

$$\vec{A} = \frac{\mu_0 I}{4\pi} \int_C \frac{d\vec{l}'}{r''}$$

From before

$$\frac{1}{r''} = \frac{1}{r} \sum_n \left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

where α is the angle between $\vec{r}$ and $\vec{r}'$.

$$\vec{A} = \frac{\mu_0 I}{4\pi} \sum_n \frac{1}{r^{n+1}} \int_C (r')^n P_n(\cos\alpha)\, d\vec{l}'$$

Monopole Term $(n=0)$

$$\vec{A}_0 = \frac{\mu_0 I}{4\pi r} \oint_C d\vec{l}' \qquad (P_0 = 1)$$

If current path is closed, $\oint_C d\vec{l}' = 0$,
so the monopole term is zero.

Dipole Term (n=1)

$$\vec{A}_1 = \frac{\mu_0 I}{4\pi r^2} \oint_c r' \cos\alpha' \, d\vec{l}'$$

$$(P_1 = x)$$

As before,

$$\hat{r} \cdot \vec{r}' = r' \cos\alpha$$

so

$$\vec{A}_1 = \frac{\mu_0 I}{4\pi r^2} \oint_c (\hat{r} \cdot \vec{r}') \, d\vec{l}'$$

Another vector identity (sometimes you need something beyond Schwamm's)

$$\oint_C \vec{H} \cdot \vec{r} \, d\vec{l} = \vec{a} \times \vec{H}$$

where $\vec{H}$ is any fixed vector and

$$\vec{a} \equiv \int_S \hat{n} \, da = \int_S d\vec{a}$$

where S is a surface bounded by C. If the surface is flat, $\vec{a} = A\hat{n}$ where A is the area of the surface.

So

$$\oint_c (\hat{r} \cdot \vec{r}\,')\, d\vec{l}\,' = \vec{a} \times \hat{r}$$

$$= \left(\int_s \hat{n}\, da \right) \times \hat{r}$$

$$\vec{A}_1 = \frac{\mu_0}{4\pi r^2} \left(I \int_s \hat{n}\, da \right) \times \hat{r}$$

<u>Dfn Magnetic Dipole Moment</u> ($\vec{m}$)

$$\vec{m} = I \int_s \hat{n}\, da$$

$$\vec{A}_1 = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$$

<u>Magnetic Dipole Field</u> (assume $\vec{m} = m\hat{y}$)

$$\vec{B}_{d.p} = \nabla \times \vec{A}_1 = \frac{\mu_0 m}{4\pi r^3} \left(2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta} \right)$$

<u>x-axis</u>

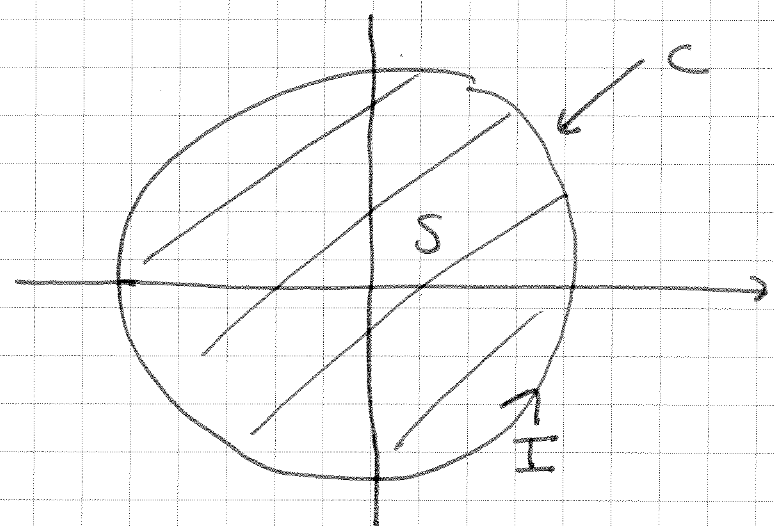
$$\vec{B} = \frac{-\mu_0 m}{4\pi |x|^3} \hat{y}$$

<u>y-axis</u>

$$\vec{B} = \frac{2\mu_0 m}{4\pi |y|^3} \hat{y}$$

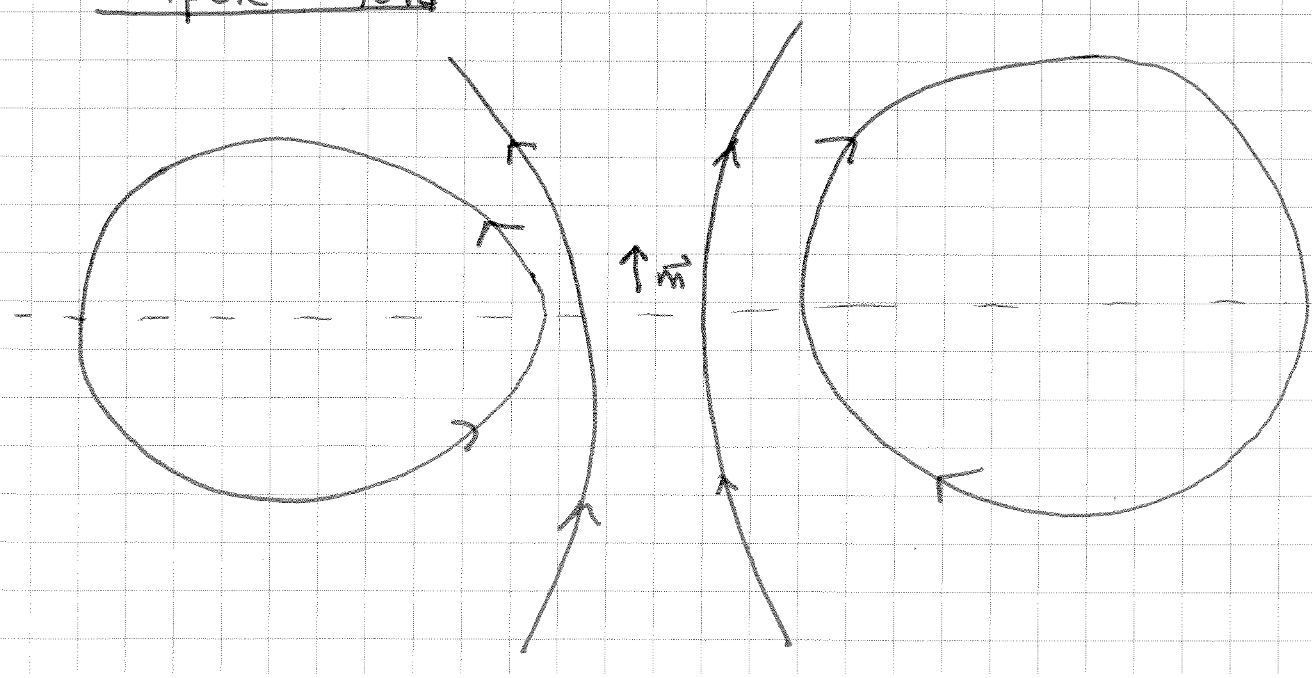
Right-Hand Rule for Moment - Curl fingers of right hand in the direction of I, thumb points toward $\vec{m}$

$\Longrightarrow$ Direct result of convention for positive surface normals.

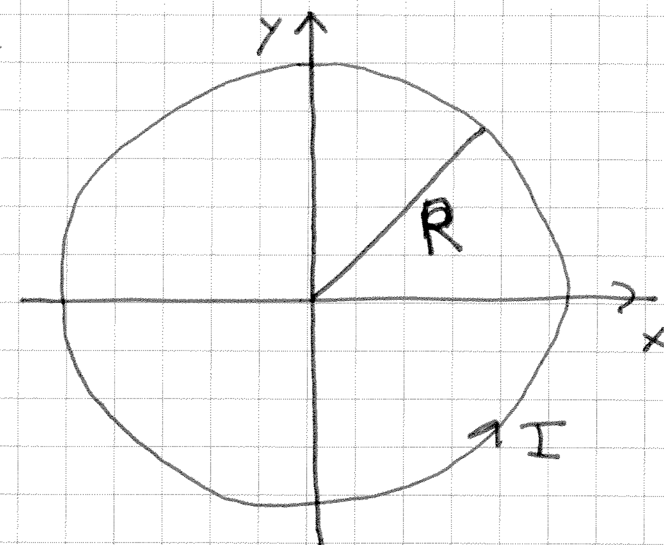


Moment out of page

Dipole Field



Ex Magnetic Moment Current Loop in x-y plane of radius R.



z out of page

As before, $\vec{m} = m\,\hat{z}$ by RHR

$$m = \pi R^2 I$$

$$\vec{m} = \pi R^2 I \,\hat{z}$$

In general, $|\vec{m}| = N\,I\,A$

turns   area

Field along the axis of the loop,

$$\vec{B} = \frac{2 m \mu_0}{4\pi z^3}\hat{z} = \frac{2\pi R^2 I \mu_0}{4\pi z^3}\hat{z}$$

$$= \frac{I R^2 \mu_0}{2 z^3}\hat{z}$$

We know the exact field along the axis from a previous calculation.

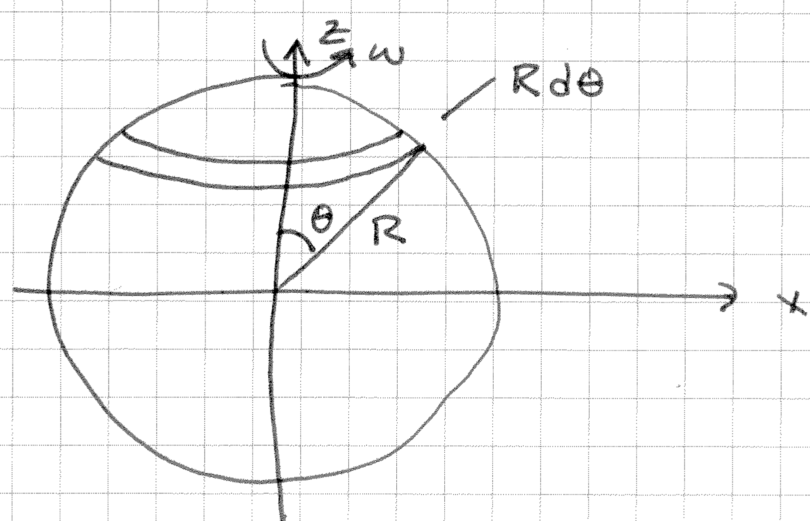
$$\vec{B}(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}} \hat{z}$$

$$= \frac{\mu_0 I R^2}{2 z^3} \left(1 + \left(\frac{R}{z}\right)^2 \right)^{-3/2} \hat{z}$$

As $z \to \infty$, $( ) \to 1$

$$\vec{B}(z) = \frac{\mu_0 I R^2}{2 z^3} \hat{z} \quad \checkmark$$

__Ex__ Compute magnetic dipole moment of thin shell with uniform surface charge density σ spinning with angular velocity ω



As before, $\vec{v} = R\sin\theta\,\omega\,\hat{\phi}$

$$\vec{K} = \sigma\vec{v} = \sigma R\sin\theta\,\omega\,\hat{\phi}$$

Cut the surface into thin strips of width $R\,d\theta$

The current in each strip is $d\vec{I} = \vec{K}\,R\,d\theta$

$$d\vec{I} = \sigma\omega R^2 \sin\theta\, d\theta\,\hat{\phi}$$

The magnetic moment of each strip is

$$dm = (dI)(A) = (dI)\,\pi(R\sin\theta)^2$$

$$dm = \pi\sigma\omega R^4 \sin^3\theta\, d\theta$$

Add the moments,

$$\vec{m} = \hat{z}\int_0^{\pi} dm = \hat{z}\int_0^{\pi} \pi\sigma\omega R^4 \sin^3\theta\, d\theta$$

$$= \hat{z}\,\pi\sigma\omega R^4 \underbrace{\int_0^{\pi} \sin^3\theta\, d\theta}_{=\,4/3}$$

$$= \frac{4}{3}\pi\sigma\omega R^4\,\hat{z}$$