Maxwell's Equations

In free space, the complete form of Maxwell's equations are:

Gauss $\quad \nabla \cdot \vec{E} = \rho/\varepsilon_0$

No Magnetic Monopoles $\quad \nabla \cdot \vec{B} = 0$

Faraday $\quad \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

Ampere $\quad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

The sources ρ, $\vec{J}$ are the total charge and current densities. Maxwell's equations also apply in matter if the charge and current densities produced by the polarization and magnetization of the materials are included.

In materials, the charge density can be separated into a bound charge density ρ_b produced by polarization and any other charge density not produced by polarization, the free charge density ρ_f.

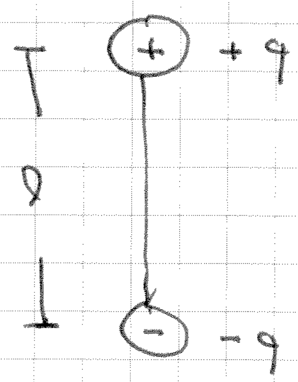
$$\rho = \rho_f + \rho_b$$

The current density $\vec{J}$ can be factored into a bound current density $\vec{J}_b$ produced by the magnetization, a current density $\vec{J}_p$ produced when the polarization is changed, and other currents called free currents.

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p$$

Polarization Current ($\vec{J}_p$)

Consider a model of the polarization as the dipole moment density ρ (note change in meaning) of stick dipoles.



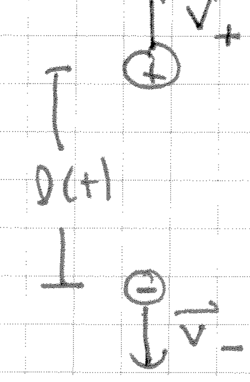
The polarization P in this model is

$$P = \rho q l$$

The time rate of change of the polarization is

$$\frac{dP}{dt} = \rho q \frac{dl}{dt}$$

The change in l causes the top and bottom charge to move



$$\cancel{\nabla} \quad v_+ = \frac{1}{2}\frac{dl}{dt} \qquad v_- = -\frac{1}{2}\frac{dl}{dt}$$

The current produced by the change in polarization in this model is

$$|\vec{J}_p| = \rho q_+ v_+ + \rho q_- v_-$$

$$= \frac{1}{2}\rho q \frac{dl}{dt} + \frac{1}{2}\rho q \frac{dl}{dt}$$

$$= \rho q \frac{dl}{dt} = \frac{dP}{dt}$$

<u>Polarization Current</u> Current resulting from changes in polarization

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$

With the division into bound and free currents we can re-write Maxwell's Eqn.

<u>Gauss</u> $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = \frac{1}{\varepsilon_0}(\rho_f + \rho_b)$

$$= \frac{1}{\varepsilon_0}(\rho_f - \nabla \cdot \vec{P})$$

No Magnetic Monopoles $\qquad \nabla \cdot \vec{B} = 0$

Faraday's Law $\qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Ampere's Law $\qquad \nabla \times \vec{B} = \mu_0 (\vec{J} + \vec{J}_d)$

$$\nabla \times \vec{B} = \mu_0 (\vec{J}_f + \vec{J}_b + \vec{J}_p + \vec{J}_d)$$

$$= \mu_0 \left(\vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

Rearrange Gauss' Law

$$\varepsilon_0 \nabla \cdot \vec{E} = \rho_f - \nabla \cdot \vec{P}$$

$$\nabla \cdot (\varepsilon_0 \vec{E} + \vec{P}) = \rho_f$$

$$\nabla \cdot \vec{D} = \rho_f$$

with $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ as before

Rearrange Ampere

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\frac{\nabla \times \vec{B}}{\mu_0} - \nabla \times \vec{M} = \vec{J}_f + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t}$$

$$\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M} \quad \text{as before}$$

$$\nabla \times \vec{H} = J_f + \frac{\partial \vec{D}}{\partial t}$$

Maxwell's Equations in Matter

$$\nabla \cdot \vec{D} = \rho_f \qquad \oint \vec{D} \cdot d\vec{a} = Q_{fenc}$$

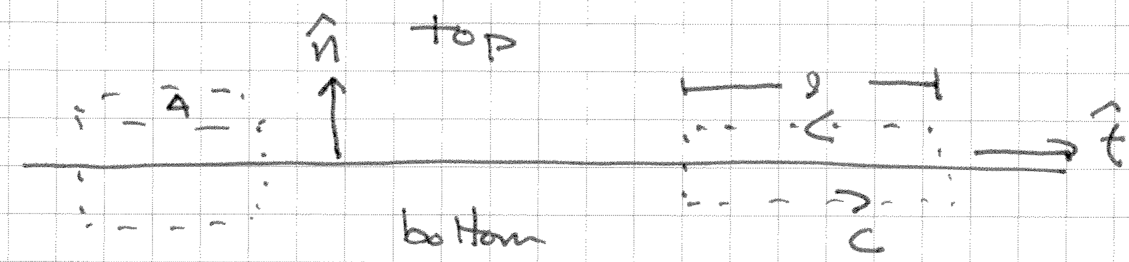
$$\nabla \cdot \vec{B} = 0 \qquad \oint \vec{B} \cdot d\vec{a} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \oint_c \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{a}$$

$$\nabla \times \vec{H} = J_f + \frac{\partial \vec{D}}{\partial t} \qquad \oint_c \vec{H} \cdot d\vec{l} = I_{fenc} + \frac{d}{dt} \int_S \vec{D} \cdot d\vec{a}$$

Boundary Conditions

Apply Maxwell's equations to a Gaussian pillbox and an Amperian path ~~between~~ at the interface between two regions.



Gauss

$$\vec{D}_t \cdot \hat{n} - \vec{D}_b \cdot \hat{n} = \sigma_f$$

$$D_t^{\perp} - D_b^{\perp} = \sigma_f$$

No Monopoles

$$\vec{B}_t \cdot \hat{n} - \vec{B}_b \cdot \hat{n} = 0$$

$$B_t^{\perp} = B_b^{\perp}$$

Faraday

$$-\vec{E}_t \cdot \hat{t} + \vec{E}_b \cdot \hat{t} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{a} \Rightarrow 0$$

if we let the area of the Amperian path go to zero.

$$E_t^{\parallel} = E_b^{\parallel}$$

Ampere

$$-\vec{H}_t \cdot \hat{t} + \hat{H}_b \cdot \hat{t} = \frac{I_{fenc}}{l} + \frac{1}{l}\frac{d}{dt}\underbrace{\int_S \vec{D} \cdot d\vec{a}}_{\substack{= \\ 0 \\ \text{as above}}}$$

$$I_{fenc} = K_f \cdot (\hat{t} \times \hat{n})\, l$$

Positive normal of path C

$$\vec{K}_f \cdot (\hat{t} \times \hat{n}) = \hat{t} \cdot (\hat{n} \times \vec{K}_f)$$ Triple Product identity

$$-\vec{H}_t \cdot \hat{t} + \vec{H}_b \cdot \hat{t} = (\hat{n} \times \vec{K}_f) \cdot \hat{t} = -(\vec{K}_f \times \hat{n}) \cdot \hat{t}$$

$$\vec{H}_t^{\parallel} - \vec{H}_b^{\parallel} = \cancel{\hat{n} \times \vec{K}_f}\ \vec{K}_f \times \hat{n}$$

If the material is a linear dielectric and magnetic material

$$\vec{D} = \varepsilon_r \varepsilon_0 \vec{E}$$

$$\mu_0 \mu_r \vec{H} = \vec{B}$$

The boundary conditions become

$$\varepsilon_{rt} \varepsilon_0 E_t^{\perp} - \varepsilon_{rb} \varepsilon_0 E_b^{\perp} = \sigma_f$$

$$B_t^{\perp} = B_b^{\perp}$$

$$E_t^{\parallel} = E_b^{\parallel}$$

$$\frac{B_t^{\parallel}}{\mu_{rt}\mu_0} - \frac{B_b^{\parallel}}{\mu_{rb}\mu_0} = \vec{K}_f \times \hat{n}$$