

Practice Final Exam - Spring 2013

Work four of the six problems. Place the problems in the order you wish them graded. The first two problems form the first half test; the second two problems form the second half test.

Problem 4.1 A wire of length 2ℓ runs along the x -axis and is centered at the origin. The wire is thinner in the middle than at the two ends. The cross-sectional area of the wire is given by $A(x) = A_0(a^2 + x^2)$, where A_0 and a are constants. Make the approximation that the current density depends only on x .

Problem 4.2 The $x - y$ plane is the boundary between two regions $z > 0$ and $z < 0$ with different electric and magnetic fields. The fields below the plane are

$$\vec{E}_- = \gamma(\hat{x} + 2\hat{y})$$

and

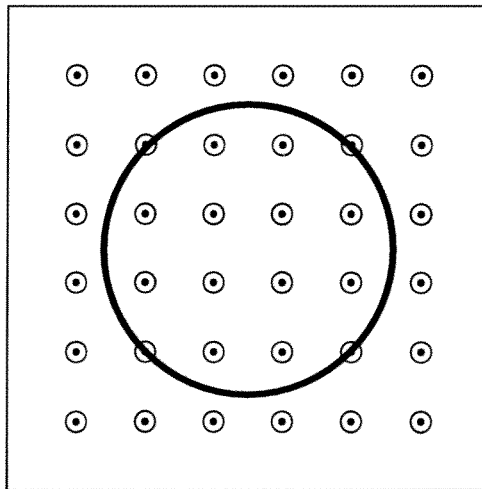
$$\vec{B}_- = \alpha(\hat{x} + \hat{z})$$

The plane has a surface charge density σ and a surface current density $\vec{K} = \Gamma\hat{y}$. Γ , γ , and α are constants. Find the electric and magnetic field above the plane ($z > 0$). Note, while this problem is electro/magnetostatic, it could be altered to include time dependent terms.

Problem 4.3 A solenoid of radius b has N turns wound over ℓ distance and carries current I . The solenoid contains a hollow cylindrical iron core with relative permeability μ_r at the operating field. The iron core has inner radius a and outer radius b . Compute \vec{B} , \vec{H} , \vec{M} , \vec{J}_b , and \vec{K}_b everywhere.

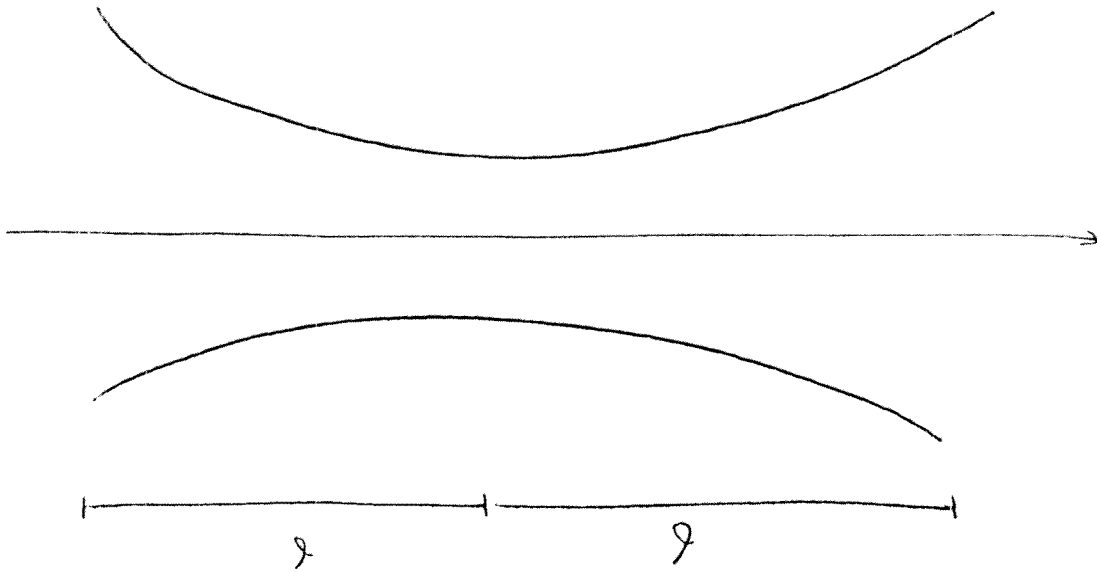
Problem 4.4 A spherical system of radius a has potential $V(a, \theta) = \gamma \cos(\theta)$ at its surface. Compute the electric field inside the system.

Problem 4.5 A circular ring of conducting wire is in a region with changing magnetic field as shown below. The radius of the ring is increasing as $a(t) = a_1 t^2$, where a_1 is constant. The magnetic field is $B(t) = B_0 t^2$ in the direction drawn with B_0 constant. The conductivity of the wire is σ and it has cross-sectional area A_w . Compute the current flowing through the wire as a function of time and give its direction.



Problem 4.6 A spherical system is built with a free charge density $\rho(r) = \gamma r$ ($r < a$) embedded in a linear dielectric material of radius a with relative permittivity ϵ_r . Calculate the potential difference between the center of the system and the surface ΔV_{0a} . γ is constant.

①



Assume a current I flows through the wire.

The current density is

$$J = \frac{I}{A}$$

This is related to the field by

$$J = \sigma E$$

$$E = \frac{\sigma I}{A}$$

The potential difference along the wires is

$$\Delta V = - \int_{-r}^r E dx = - \sigma I \int_{-r}^r \frac{dx}{A_0 + x^2 \frac{\sigma}{\epsilon_0}}$$

The potential difference across the ends of the wire is given by

$$\Delta V = - \int_{-l}^l \vec{E} \cdot d\vec{l}$$

$$= - \sigma I \int_{-l}^l \frac{dx}{A(x)}$$

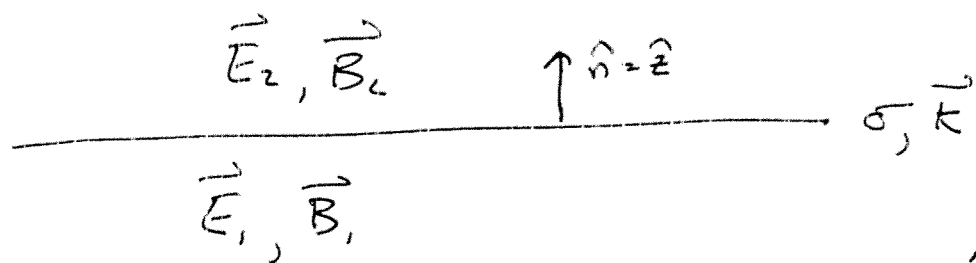
$$= - \frac{\sigma I}{A_0} \int_{-l}^l \frac{dx}{a^2 + x^2}$$

$$= - \frac{2\sigma I}{A_0} \int_0^l \frac{dx}{a^2 + x^2} = \frac{-2\sigma I}{a A_0} \tan^{-1}\left(\frac{l}{a}\right)$$

The resistance is given by Ohm's law

$$R = \frac{\Delta V}{I} = \frac{2\sigma}{a A_0} \tan^{-1}\left(\frac{l}{a}\right)$$

②



Boundary Conditions

$$E_2^\perp - E_1^\perp = \sigma / \epsilon_0$$

$$E_{2z} = E_{1z} + \sigma / \epsilon_0 = \sigma / \epsilon_0$$

$$\vec{E}_2'' = \vec{E}_1'' \quad \vec{E}_2'' = \gamma \hat{x} + 2\gamma \hat{y}$$

$$\vec{E}_2 = \gamma \hat{x} + 2\gamma \hat{y} + \frac{\sigma}{\epsilon_0} \hat{z}$$

$$B_2^\perp = B_1^\perp \Rightarrow B_2^\perp = \alpha \hat{z}$$

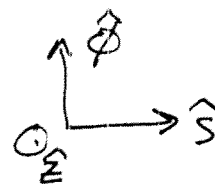
$$\vec{B}_2'' - \vec{B}_1'' = \mu_0 (\vec{K} \times \hat{n}) = \mu_0 (\Gamma \hat{y} \times \hat{z}) = \mu_0 \Gamma \hat{x}$$

$$\vec{B}_2'' = \vec{B}_1'' + \mu_0 \Gamma \hat{x} = \alpha \hat{x} + \mu_0 \Gamma \hat{x} = (\alpha + \mu_0 \Gamma) \hat{x}$$

$$\vec{B}_2 = (\alpha + \mu_0 \Gamma) \hat{x} + \alpha \hat{z}$$

4.3

$$n = \frac{N}{l}$$



The magnetic field of an infinite solenoid is

$$\vec{B}_i = \mu_0 n I \hat{z}, \quad \vec{B}_o = 0$$

ignoring the iron. Therefore, the \vec{H} field is

$$s < b, \quad \vec{H} = \frac{\vec{B}_i}{\mu_0} = n I \hat{z}$$

$$s > b, \quad \vec{H} = 0 \Rightarrow \vec{B} = 0 \Rightarrow \vec{M} = 0$$

The introduction of the iron does not change the field.

$$\text{For } s < a, \quad \vec{M} = 0$$

$$\vec{B} = \mu_0 \vec{H} = \mu_0 n I \hat{z}$$

$$\text{For } a < s < b,$$

$$\vec{B} = \mu_0 \mu_r \vec{H} = \mu_0 \mu_r n I \hat{z}$$

The magnetization is $\vec{M} = \chi_m \vec{H} = (\mu_r - 1) \vec{H}$
 $= (\mu_r - 1) n I \hat{z}$

The bound current density

$$\vec{J}_b = \nabla \times \vec{M} = 0$$

Surface current density, $s = a$ surface

$$\hat{n} = -\hat{s}$$

$$\vec{K}_b = \vec{M} \times (-\hat{s}) = -(\mu_r - 1) n I (\hat{z} \times \hat{s})$$
$$= -(\mu_r - 1) n I \hat{\phi}$$

Surface current density, $s = b$, $\hat{n} = +\hat{s}$

$$\vec{K}_b = \vec{M} \times \hat{s} = (\mu_r - 1) n I \hat{\phi}$$

Note, surface currents enhance field in iron

4.4 Since the system is spherical, the general solution of Laplace's eqn is

$$V(r, \theta) = \sum_n \left(A_n r^n + B_n r^{-(n+1)} \right) P_n(\cos \theta)$$

For the solution to be finite, $B_n = 0$.

The potential at the surface is given as

$$V(a, \theta) = \Gamma \cos \theta = \Gamma P_1(\cos \theta)$$

By orthogonality, $A_n = 0$ if $n \neq 1$.

$$V(a, \theta) = A_1 a^1 P_1(\cos \theta) = \Gamma P_1(\cos \theta)$$

$$A_1 = \frac{\Gamma}{a}$$

The potential is then

$$V(r, \theta) = \frac{\Gamma r}{a} P_1(\cos \theta) = \frac{\Gamma r}{a} \cos \theta$$

The electric field inside the sphere is

$$\vec{E} = -\nabla V$$

$$= -\left[\frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\phi} \right]$$

$$\vec{E} = -\left[\frac{\rho}{a} \cos \theta \hat{r} + \frac{1}{r} \cdot \frac{\rho r}{a} \cdot -\sin \theta \hat{\theta} + 0 \right]$$

$$= -\frac{\rho}{a} \left[\cos \theta \hat{r} - \sin \theta \hat{\theta} \right] = -\frac{\rho}{a} \hat{z}$$

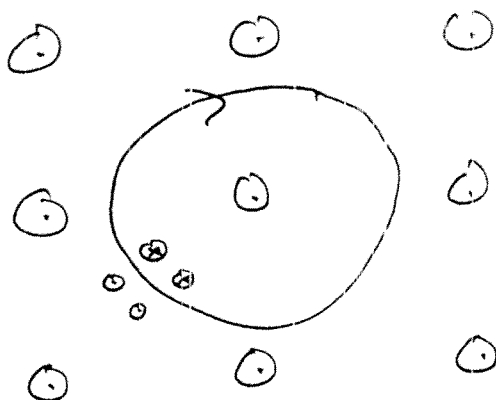
We could get this more immediately by recognizing

$$z = r \cos \theta$$

$$V = \frac{\rho}{a} z$$

$$\vec{E} = -\nabla V = -\frac{\rho}{a} \hat{z}$$

4.5 To oppose an increasing flux out of the page, the induced current must flow clockwise



The resistance of the loop is $R = \frac{2\pi a}{\sigma A_w}$

The flux through the loop is

$$\begin{aligned}\Phi_m &= BA = (B_0 t^2) \pi (a, t^2)^2 \\ &= B_0 a,^2 \pi t^6\end{aligned}$$

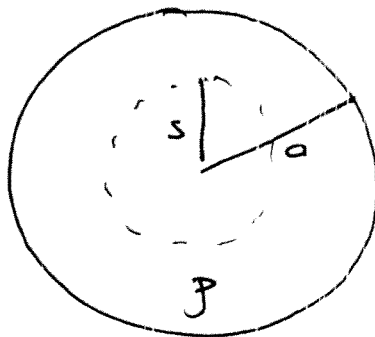
The emf around the loop is

$$\text{emf} = -\frac{d\Phi_m}{dt} = -6B_0 a,^2 \pi t^5$$

The current is then

$$I = \frac{\text{emf}}{R} = \frac{6B_0 a,^2 \pi t^5}{2\pi a,^2 / \sigma A_w} = 3\sigma A_w a,^2 t^3 B_0$$

4.6



The charge enclosed in a spherical Gaussian surface of radius r

$$Q_{\text{enc}} = \int_0^r 4\pi r^2 \rho \, dr = \int_0^r 4\pi r^2 \gamma r \, dr$$
$$= 4\pi \gamma \frac{r^4}{4} = \pi \gamma r^4$$

The displacement field is then

$$\oint \vec{D} \cdot d\vec{\sigma} = 4\pi r^2 D = Q_{\text{enc}}$$

$$\vec{D} = \frac{\pi \gamma r^4 \hat{r}}{4\pi r^2} = \frac{\gamma r^2}{4} \hat{r}$$

Since the material is linear,

$$\vec{D} = \epsilon \vec{E} = \epsilon_r \epsilon_0 \vec{E}$$

and

$$\vec{E} = \frac{\gamma r^2}{4\epsilon_r \epsilon_0} \hat{r}$$

The potential difference between the center and the surface is

$$\Delta V_{00} = - \int_{0 \rightarrow a} \vec{E} \cdot d\vec{l} = - \int_0^a \frac{\gamma r^2}{4\epsilon_r \epsilon_0} \hat{r} \cdot \hat{r} dr$$

$$= - \frac{\gamma}{4\epsilon_r \epsilon_0} \int_0^a r^2 = - \frac{\gamma}{4\epsilon_r \epsilon_0} \frac{a^3}{3}$$

$$= - \frac{\gamma a^3}{12\epsilon_r \epsilon_0}$$