

Electricity and Magnetism - Practice Test 1 - Spring 2013

Work four of the six problems. Place the problems in the order you wish them graded. The first two problems form the first half test; the second two problems form the second half test.

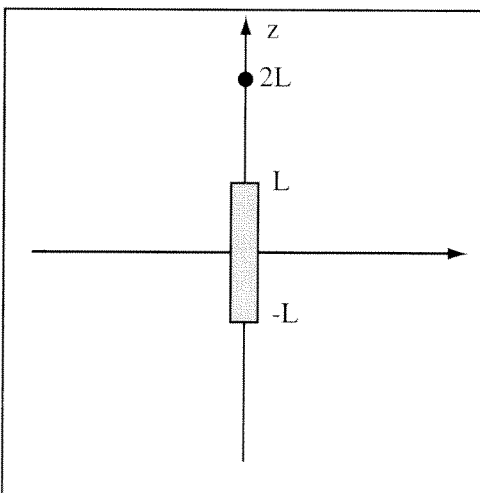
Problem 1.1 A spherical system has a NON-UNIFORM volume charge density $\rho = \gamma/r$ for $r < a$ and $\rho = 0$ for $r > 0a$ where γ is a constant. Compute either the electric potential everywhere or the total energy of the system.

Problem 1.2 Calculate the electric potential at a point $(0, 0, z)$ of a finite cylinder ($s < a$, $0 < z < \ell$) containing a uniform volume charge density ρ . Work on the problem until you have a single one-dimensional integral to do.

Problem 1.3 An infinite cylinder of radius a contains a uniform volume charge density ρ . Compute the potential difference between a point on the axis and a point on the outside surface.

Problem 1.4 Calculate the field and potential at the origin of a NON-UNIFORM spherical volume charge of radius a and charge density $\rho = \gamma \sin(2\theta)$ where γ is constant. (I would probably choose either field or potential on a test).

Problem 1.5 A finite linear charge occupies the region between $-L$ and $+L$ along the z axis and has uniform linear charge density λ . Calculate the electric field at a point a distance $2L$ along the z axis.

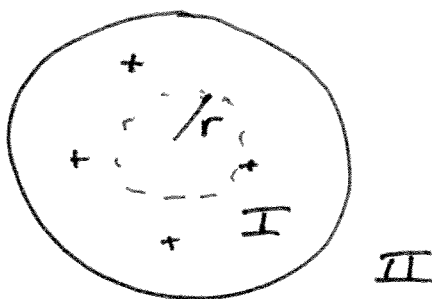


Problem 1.6 The electric potential in some region of space is $V = V_0x^2 - V_0y^2$. Compute the electric field in cylindrical coordinates. What is the charge density in the region containing the field?

1.1

A sphere has non-uniform volume charge density $\rho = \frac{\gamma}{r}$ for $r < a$ and 0 for $r > a$.

Compute the total energy of the sphere.



Spherical
Gaussian
Surface
Radius r .

Region I

$$Q_{\text{enc}} = \int_0^r 4\pi r^2 \rho dr = 4\pi\gamma \int_0^r dr r$$
$$= 4\pi\gamma \frac{r^2}{2} = 2\pi\gamma r^2$$

$$\vec{E}_I = \frac{Q_{\text{enc}}}{4\pi\epsilon_0 r^2} \hat{r} = \frac{2\pi\gamma r^2}{4\pi\epsilon_0 r^2} \hat{r} = \frac{\gamma}{2\epsilon_0} \hat{r}$$

Region II

$$Q_{\text{enc}} = 2\pi\gamma a^2$$

$$\vec{E}_{II} = \frac{2\pi\gamma a^2}{4\pi\epsilon_0 r^2} \hat{r} = \frac{\gamma a^2}{2\epsilon_0 r^2} \hat{r}$$

Potential

$$V_I = - \int E_I dr = -\frac{\gamma r}{2\epsilon_0} + C_I$$

$$V_{II} = - \int E_{II} dr = \frac{\gamma a^2}{2\epsilon_0 r} + C_{II}$$

$$V_{II}(\infty) = 0 \Rightarrow C_{II} = 0$$

Continuity

$$V_I(a) = V_{II}(a)$$

$$-\frac{\gamma a}{2\epsilon_0} + C_I = \frac{\gamma a^2}{2\epsilon_0 a} = \frac{\gamma a}{2\epsilon_0}$$

$$C_I = \frac{\gamma a}{\epsilon_0}$$

$$V_I = -\frac{\gamma r}{2\epsilon_0} + \frac{\gamma a}{\epsilon_0}$$

$$V_{II} = \frac{\gamma a^2}{2\epsilon_0 r}$$

Total Energy

$$U = \int_{\text{space}} \frac{1}{2} \epsilon_0 E^2 d\tau$$

$$= \int_0^a \frac{1}{2} \epsilon_0 E^2 d\tau + \int_a^{\infty} \frac{1}{2} \epsilon_0 E^2 d\tau$$

Inside sphere (U_I)

$$U_I = \frac{1}{2} \epsilon_0 E^2 = \frac{1}{2} \epsilon_0 \left(\frac{\gamma}{2\epsilon_0} \right)^2$$
$$= \frac{1}{8} \frac{\gamma^2}{\epsilon_0}$$

Since the energy density is constant, the total energy is the energy density multiplied by the volume.

$$U_I = u_I V = \frac{1}{8} \frac{\gamma^2}{\epsilon_0} \cdot \frac{4}{3} \pi a^3$$
$$= \frac{1}{6} \frac{\pi \gamma^2 a^3}{\epsilon_0}$$

The energy density outside is

$$U_{II} = \frac{1}{2} \epsilon_0 E^2 = \frac{1}{2} \epsilon_0 \left(\frac{Q^2}{4\pi\epsilon_0 r^2} \right)^2$$

$$= \frac{1}{32} \frac{Q^2}{\pi^2 \epsilon_0 r^4}$$

where $Q = 2\pi\gamma a^2$ the total charge of the sphere.

$$U_{II} = \int_a^\infty 4\pi r^2 U_{II} dr$$

angular variables
integrated out.

$$= \int_a^\infty 4\pi \frac{Q^2}{32\pi^2 \epsilon_0 r^2} dr$$

$$= \frac{1}{8} \frac{Q^2}{\pi \epsilon_0} \left(-\frac{1}{r} \right)_a^\infty$$

$$= \frac{1}{8} \frac{Q^2}{\pi \epsilon_0 a} = \frac{1}{8} \frac{(2\pi\gamma a^2)^2}{\pi \epsilon_0 a}$$

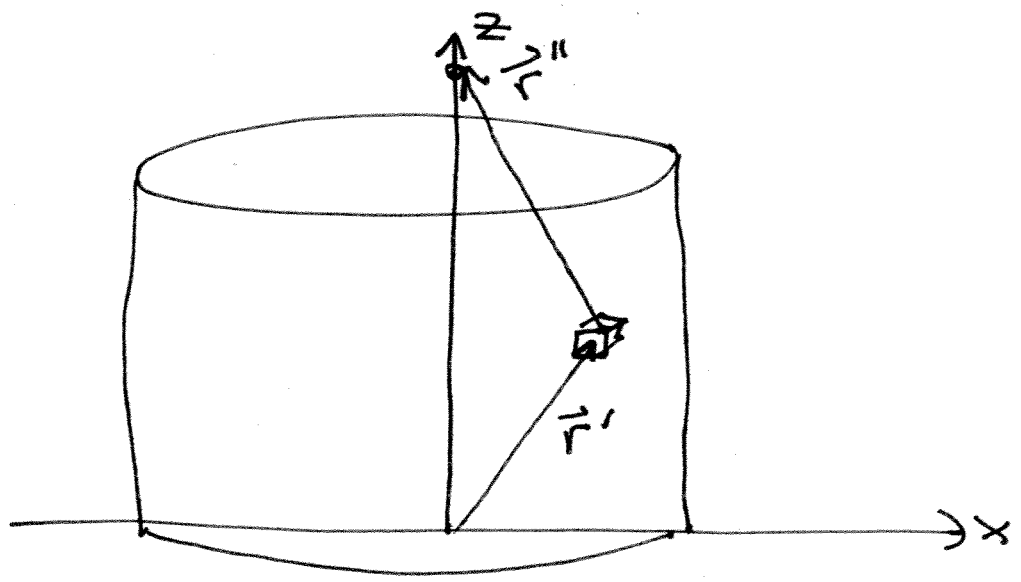
$$= \frac{1}{2} \frac{\pi \gamma^2 a^3}{\epsilon_0}$$

Total Energy

$$U = U_I + U_{II} = \frac{1}{2} \pi \frac{\gamma^2 d^3}{\epsilon_0} + \frac{1}{6} \pi \frac{\gamma^2 d^3}{\epsilon_0}$$
$$= \frac{2}{3} \pi \frac{\gamma^2 d^3}{\epsilon_0}$$

1.2

Calculate the electric potential of a finite cylinder ($s < a$, $0 < z < l$) containing a uniform charge density ρ at a point $z > l$ along the z -axis.



Field Point $\vec{r} = (0, 0, z)$

Source Point $\vec{r}' = s' \hat{s}' + z' \hat{z}$

Displacement $\vec{r}'' = \vec{r} - \vec{r}'$
 $= -s' \hat{s}' + (z - z') \hat{z}$

Length $r'' = \sqrt{s'^2 + (z - z')^2}$

Electric Potential

$$V(z) = \int_{\text{cylinder}} \frac{\rho d\tau'}{4\pi\epsilon_0 r''}$$

$$d\tau' = (ds')(s'd\phi') dz'$$

$$V(z) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^l dz' \int_0^a ds' \frac{\rho s'}{\sqrt{s'^2 + (z-z')^2}}$$

Integrate $d\phi'$ to give 2π

$$V(z) = \frac{\rho}{2\epsilon_0} \int_0^l dz' \int_0^a ds' \frac{s'}{\sqrt{s'^2 + (z-z')^2}}$$

Integrate s'

$$\int_0^a ds' \frac{s'}{\sqrt{s'^2 + (z-z')^2}} = \sqrt{s'^2 + (z-z')^2} \Big|_0^a$$

$$= \sqrt{a^2 + (z-z')^2} \quad (\text{Skhroums})$$

$$- |z-z'|$$

$$z-z' > 0$$

$$= \sqrt{a^2 + (z-z')^2} - (z-z')$$

$$V(z) = \frac{\rho}{2\epsilon_0} \int_0^l dz' \left(\sqrt{a^2 + (z-z')^2} - (z-z') \right)$$

That's as far as the problem asked you to take it, but let's complete it.

$$\int_0^l dz' (z-z') = zl - \frac{l^2}{2}$$

$$\int_0^l \sqrt{a^2 + (z-z')^2} dz' = \int_0^l \sqrt{a^2 + u^2} du \quad \begin{array}{l} u = z - z' \\ du = -dz' \end{array}$$

$$= - \int_0^{z-l} \sqrt{a^2 + u^2} du$$

$$\int_z^{z-l} du \sqrt{a^2 + u^2} = \left(\frac{1}{2} u \sqrt{u^2 + a^2} + \frac{a^2}{2} \ln(u + \sqrt{u^2 + a^2}) \right) \Big|_z^{z-l}$$

$$= \frac{1}{2} (z-l) \sqrt{(z-l)^2 + a^2} - \frac{1}{2} z \sqrt{z^2 + a^2}$$

$$+ \frac{a^2}{2} \ln \left(z-l + \sqrt{(z-l)^2 + a^2} \right) - \frac{a^2}{2} \ln \left(z + \sqrt{z^2 + a^2} \right)$$

$$= \frac{1}{2} (z-l) \sqrt{(z-l)^2 + a^2} - \frac{1}{2} z \sqrt{z^2 + a^2}$$

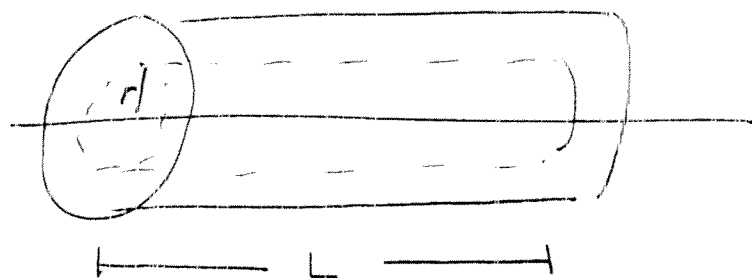
$$+ \frac{a^2}{2} \ln \left(\frac{z-l + \sqrt{(z-l)^2 + a^2}}{z + \sqrt{z^2 + a^2}} \right)$$

So the total potential is

$$V(z) = \frac{q}{2\epsilon_0} \left[\frac{1}{2} (z-d) \sqrt{(z-d)^2 + a^2} - \frac{1}{2} z \sqrt{z^2 + a^2} + \frac{a^2}{2} \ln \left(\frac{z-d + \sqrt{(z-d)^2 + a^2}}{z + \sqrt{z^2 + a^2}} \right) + \frac{d^2}{2} - zd \right]$$

At least the units are correct; I don't feel like checking the $z \rightarrow \infty$ limit.

1.3



Use a cylindrical Gaussian surface of radius r and length L .

The charge inside the surface is $Q_{\text{enc}} = \rho V$
 $= \rho \pi r^2 L$

The flux out of the cylinder is

$$\Phi = EA_s = 2\pi r L E = \frac{Q_{\text{enc}}}{\epsilon_0} \quad +5$$

$$E = \frac{Q_{\text{enc}}}{2\pi r L \epsilon_0} = \frac{\rho r^2 L}{2\pi r L \epsilon_0} = \frac{r \rho}{2\epsilon_0} \quad +5$$

The potential difference between $r=0$ and $r=a$ is

$$\Delta V_{0a} = - \int_0^a E dr = - \int_0^a \frac{\rho r}{2\epsilon_0} dr = - \frac{\rho a^2}{4\epsilon_0} \quad +5$$

where the sign is correct because the potential decreases in the direction of the field.

1.4

Calculate the field and potential at the origin of a NON-UNIFORM spherical volume charge of radius a and charge density

$$\rho = \gamma \sin 2\theta$$

where γ is a constant

Compute Field

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_{\text{volume}} \frac{\rho d\tau}{r'^2} \hat{r}''$$

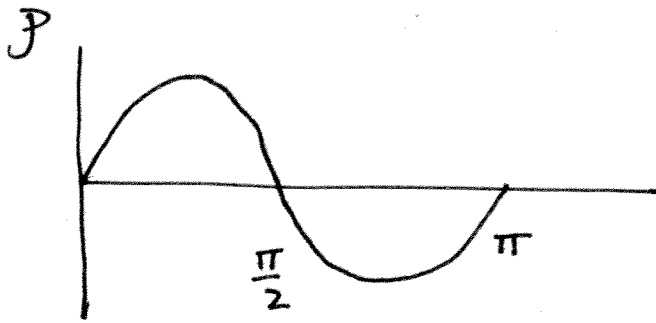
$\vec{r} = 0$ Field point

$\vec{r}' = r' \hat{r}'$ Source Point

$$\vec{r}'' = \vec{r} - \vec{r}' = -r' \hat{r}' \quad r'' = r'$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^\pi d\theta' \int_0^a \frac{\gamma \sin 2\theta'}{r'^2} r'^2 \sin\theta' \hat{r}'$$

$$\hat{r}' = \cos\theta \hat{z} \quad (\text{Discarding } \hat{x}, \hat{y} \text{ which are zero by symmetry})$$



Field should point in \hat{z} direction

$$\vec{E} = -\frac{\gamma}{4\pi\epsilon_0} \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_0^a dr' \sin 2\theta' \sin \theta' \cos \theta' \hat{z}$$

$$= \frac{-2\pi a \gamma}{4\pi\epsilon_0} \int_0^\pi d\theta' \sin 2\theta' \sin \theta' \cos \theta' \hat{z}$$

$$= \frac{-a\gamma}{4\epsilon_0} \int_0^\pi \sin^2 2\theta' d\theta' \hat{z}$$

$$\frac{1}{2} \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$$

~~$$\int_0^{2\pi} \sin 2\theta d\theta = \frac{\pi}{2}$$~~

~~$$\vec{E} = \frac{-a\pi\gamma}{4\epsilon_0} \hat{z}$$~~

$$\int_0^{\pi} \sin^2 2\theta' d\theta' = \frac{1}{2} \int_0^{2\pi} \sin^2 u du$$

$$u = 2\theta' \quad du = 2d\theta'$$

$$\int_0^{2\pi} \sin^2 \theta d\theta = \frac{1}{2} \int_0^{2\pi} \sin^2 \theta + \cos^2 \theta d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} d\theta = \pi$$

$$\int_0^{\pi} \sin^2 2\theta d\theta = \frac{\pi}{2}$$

$$\vec{E} = \frac{-\sigma \gamma}{4\epsilon_0} \frac{\pi}{2} \hat{z}$$

$$= \frac{-\sigma \pi \gamma}{8\epsilon_0} \hat{z}$$

Check Units

$$[\rho] = C/m^3 = [\gamma]$$

$$[E] = \left[\frac{C/m^2}{\epsilon_0} \right] \checkmark$$

Compute Potential at Origin Should be zero.

$$V(0) = \int \frac{\rho d\tau'}{4\pi\epsilon_0 r'}$$

$$= \int_0^a dr' \int_0^{2\pi} d\phi' \int_0^\pi d\theta' \frac{r'^2 \sin\theta' \gamma \sin 2\theta'}{4\pi\epsilon_0 r'}$$

$$= \frac{\gamma}{4\pi\epsilon_0} \left[\int_0^{2\pi} d\phi' \right] \left[\int_0^a r' dr' \right] \left[\int_0^\pi \sin\theta' \sin 2\theta' d\theta' \right]$$

" " "

2π a

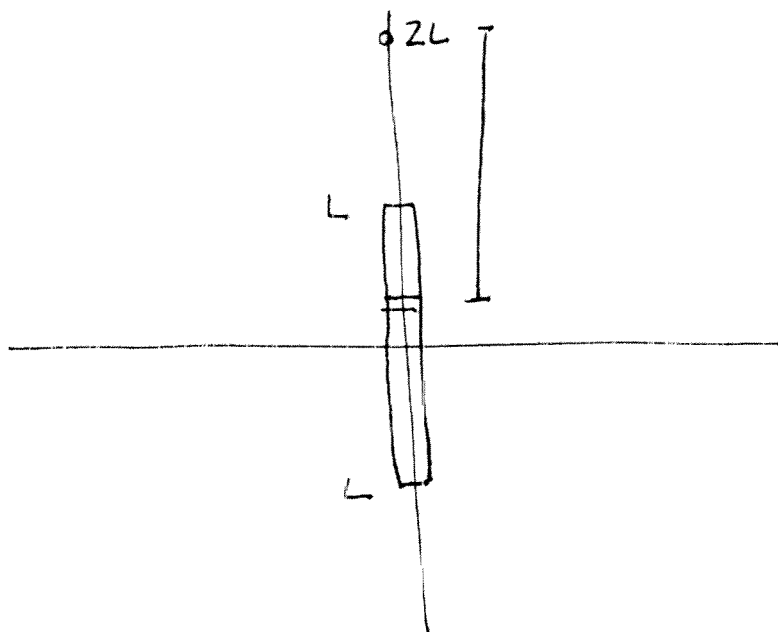
$$\int_0^\pi \sin\theta' \sin 2\theta' d\theta' = \int_0^\pi 2 \sin^2\theta' \cos\theta' d\theta'$$

$$u = \sin\theta' \quad du = \cos\theta' d\theta'$$

$$= 2 \int u^2 du = \frac{2}{3} u^3$$

$$= \frac{2}{3} \sin^3\theta' \Big|_0^\pi = 0 \quad \checkmark$$

1.5



$$d = 2L - z$$

Coulomb's Law

$$\vec{E} = \int_{-L}^L \frac{k dq \hat{r}''}{r''^2} \quad +5$$

$$\hat{r}'' = \hat{z}$$

$$r'' = d$$

$$dq = \lambda dz$$

$$\vec{E} = \hat{z} \int_{-L}^L \frac{k \lambda dz}{(2L - z)^2} \quad +10$$

$$u = 2L - z \quad du = -dz$$

$$\vec{E} = -\hat{z} \int_{3L}^L \frac{k \lambda du}{u^2} = k \lambda \hat{z} \left(\frac{1}{u} \right)_{3L}^L$$

$$\vec{F} = \kappa \lambda \hat{z} \left(\frac{1}{L} - \frac{1}{3L} \right)$$

$$= \frac{2}{3} \frac{\kappa \lambda}{L} \hat{z} \quad +10$$

1.6

$$V = V_0(x^2 - y^2)$$

$$= V_0(s^2 \cos^2 \phi - s^2 \sin^2 \phi)$$

$$= V_0 s^2 (\cos^2 \phi - \sin^2 \phi)$$

$$= V_0 s^2 \cos 2\phi \quad (\text{Trig identity wiki})$$

Electric field

$$\vec{E} = -\nabla V = -V_0 \left(\frac{\partial s^2 \cos 2\phi}{\partial s} \hat{s} + \frac{1}{s} \frac{\partial s^2 \cos 2\phi}{\partial \phi} \hat{\phi} \right)$$

$$= -2V_0 s \cos 2\phi \hat{s} + 2V_0 s \sin 2\phi \hat{\phi} \quad +95$$

Charge Density $\nabla \cdot \vec{E} = \rho/\epsilon_0$

$$\rho = \epsilon_0 \nabla \cdot \vec{E}$$

$$= 2V_0 \epsilon_0 \left[\frac{1}{s} \frac{\partial}{\partial s} (-s^2 \cos 2\phi) + \frac{1}{s} \frac{\partial}{\partial \phi} (s \sin 2\phi) \right]$$

$$= 2V_0 \epsilon_0 \left[-2 \cos 2\phi + 2 \cos 2\phi \right] = 0$$

+10

Try Cartesian

$$-\nabla V = \vec{E} = -2V_0(x\hat{x} - y\hat{y})$$

Charge

$$\epsilon_0 \nabla \cdot \vec{E} = -2V_0 \epsilon_0 (1 - 1) = 0$$