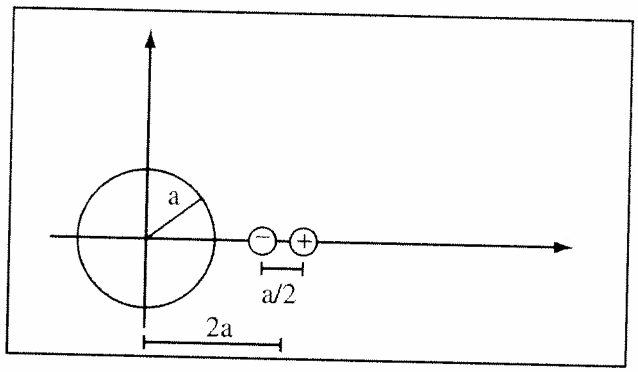
Electricity and Magnetism - Practice Test 2

Work four of the six problems. Place the problems in the order you wish them graded. The first two problems form the first half test: the second two problems form the second half test.

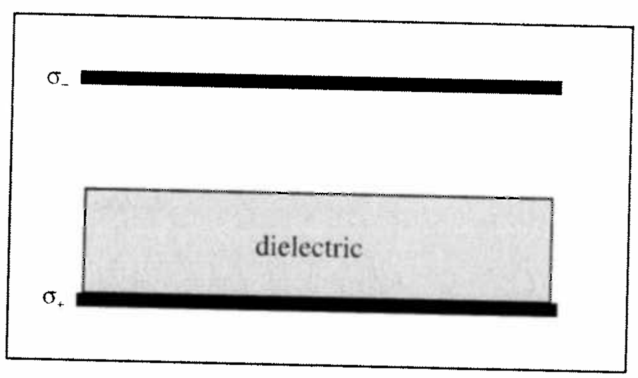
Problem 2.1 A spherical object with radius a has a potential at its surface that has value V_0 for a small patch with $0 < \theta < \pi/8$ at its north pole. The potential of the rest of the object is 0. Compute first two non-zero terms of the potential inside the sphere.

Problem 2.2 A dipole is place outside of a grounded conducting sphere with radius a with its dipole moment pointing in a direction normal to the sphere, as drawn. The charge on the two ends of the dipole are $\pm q$. The center of the dipole is a distance $2a$ from the center of the sphere. The distance between the two charges of the dipole is $a/2$. Compute the force the sphere exerts on the positive charge in the dipole. (I initially wanted the force on the dipole but it was too annoying.)



Problem 2.3 A infinite conducting cylinder of radius a has a surface charge density $\sigma(\phi) = \sigma_0(\sin^2(\phi) - \frac{1}{2})$. Compute the potential outside the cylinder.

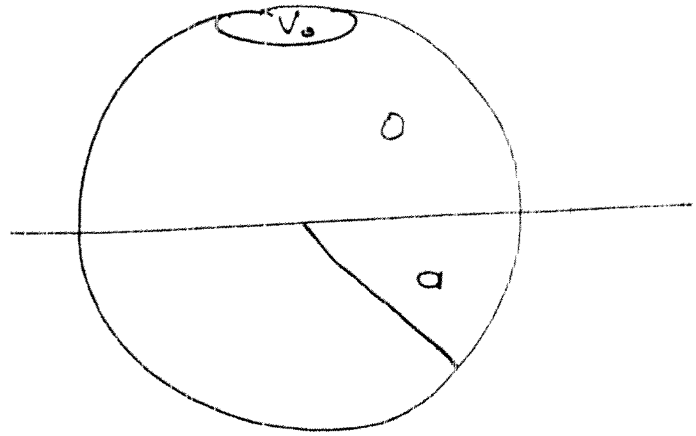
Problem 2.4 A linear dielectric slab with dielectric constant ϵ_r is placed between two infinite parallel planes of charge with charge density $\pm\sigma$. Find $\vec{D}$, $\vec{E}$, $\vec{P}$, and ρ_b in the dielectric, and the bound charge density on the top and bottom surface of the dielectric.



Problem 2.5 A potential of $V_0 \cos(\theta)$ is established on the inner surface of a spherical dielectric with inner radius a and outer radius b. The dielectric constant of the material is ϵ_r. Find the potential for $r > a$. You may report a system of equations that needs to be solved to find the coefficients of the potential functions. Actually solving these equations turns out to be quite messy. These equations should be a set of simple linear, non-differential equations.

Problem 2.6 A spherical system has polarization $\vec{P} = \gamma r^2 \hat{r}$ for radius $r < a$ and $\vec{P} = 0$ for $r > a$. Find the electric field everywhere.

2.1



$$\frac{\pi}{8} \cdot \frac{360}{2\pi} = 22.5^\circ$$

The solution to Laplace's equation in spherical coordinates, discarding solutions that are infinite at the origin.

$$V^i = \sum A_n s^n P_n(\cos\theta) \qquad + 10$$

Apply the boundary condition

$$V^i(a,\theta) = \begin{cases} V_0 & 0 < \theta < \pi/8 \\ 0 & \pi/8 < \theta < \pi \end{cases}$$

$$V^i(a,\theta) = \sum A_n a^n P_n(\cos\theta)$$

Multiply by $P_m(\cos\theta)$

$$I \equiv \int_{-1}^{1} V^i(a,\theta)\, P_m(\cos\theta)\, d(\cos\theta)$$

$$= \sum A_n a^n \underbrace{\int_{-1}^{1} P_n(\cos\theta) P_m(\cos\theta)\, d(\cos\theta)}_{\frac{2}{2m+1}\delta_{nm}}$$

$$= \frac{2 A_m a^m}{2m+1} \quad +S$$

Work on I

$$I = -\int_{\pi}^{0} V^i(a,\theta)\, P_m(\cos\theta)\, (d\theta) \sin\theta$$

$$= \int_0^{\pi} V^i(a,\theta) \sin\theta\, P_m(\cos\theta)\, d\theta$$

$$= V_0 \int_0^{\pi/8} \sin\theta\, P_m(\cos\theta)\, d\theta$$

$\underline{P_0 = 1}$

$$I_0 = V_0 \int_0^{\pi/8} \sin\theta \, d\theta = -V_0 \cos\theta \Big|_0^{\pi/8}$$

$$= V_0 \left(1 - \cos \pi/8 \right) = \frac{2 A_0}{1}$$

$$A_0 = \frac{V_0}{2} \left(1 - \cos \pi/8 \right) \qquad +5$$

$P_1 = \cos\theta$

$$I_1 = V_0 \int_0^{\pi/8} \cos\theta \sin\theta \, d\theta$$

$$u = \cos\theta$$

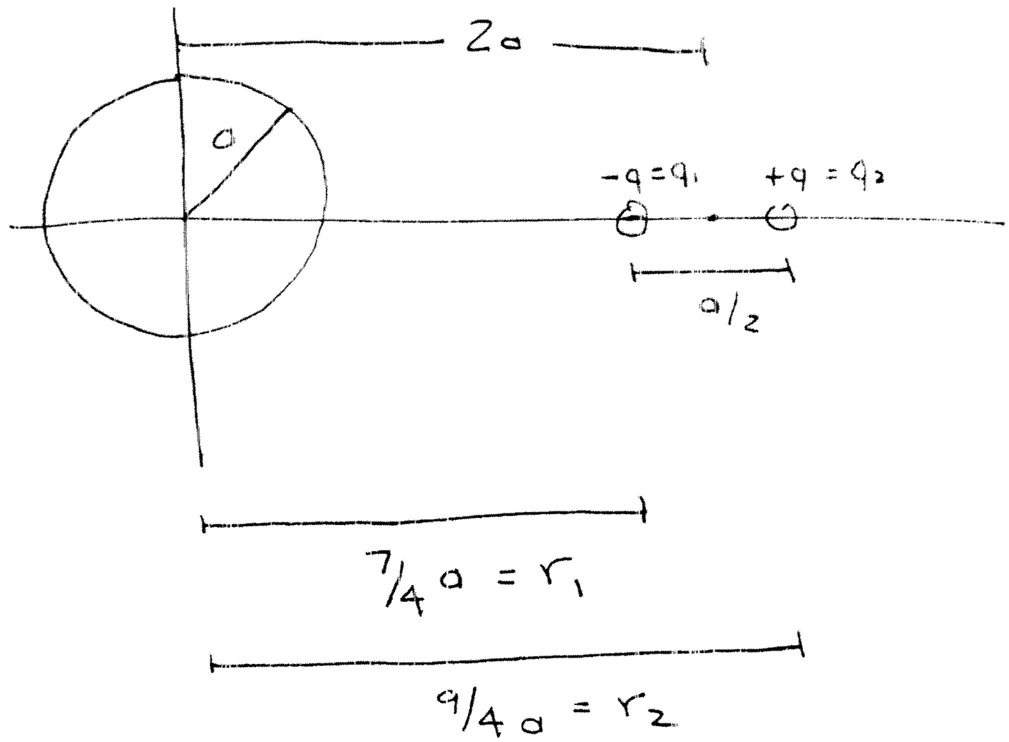
$$du = -\sin\theta \, d\theta$$

$$I_1 = -V_0 \int_1^{\cos \pi/8} u \, du = -\frac{V_0}{2} \left(\cos^2 \frac{\pi}{8} - 1 \right)$$

$$= \frac{V_0}{2} \left(1 - \cos^2 \pi/8 \right) = \frac{2 A_1 a}{3}$$

$$A_1 = \frac{3}{4} \frac{V_0}{a} \left(1 - \cos^2 \pi/8 \right) \qquad +5$$

2.2



We will require two image changes

$$q_1' = -\frac{a}{r_1} q_1 = -\frac{a}{\frac{7}{4}a}(-q) = \frac{4}{7} q$$

at

$$r_1' = \frac{a^2}{r_1} = \frac{4}{7} a$$

+7

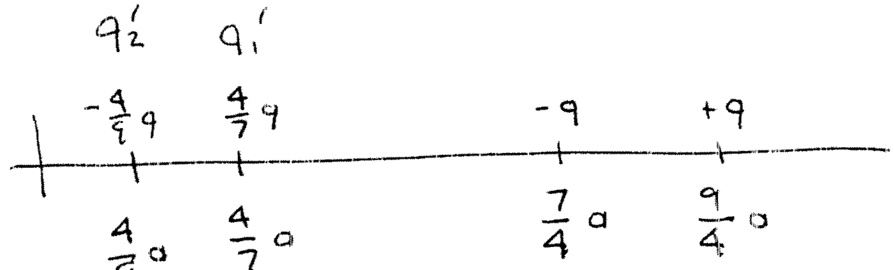
and

$$q_2' = -\frac{a}{r_2} q_2 = \frac{-a}{9/4\, a} q$$

$$= -\frac{4}{9} q$$

+7

at $\quad r_2' = \frac{a^2}{r_2} = \frac{4}{9} a$



<u>Force on $+q$</u>

$$\vec{F}_+ = \frac{k q_1' q_2}{\left(9/4\, a - \frac{4}{7} a\right)^2} + \frac{k q_2' q_2}{\left(9/4\, a - \frac{4}{9} a\right)^2}$$

+11

$$= kq \left(\frac{\frac{4}{9} q}{\left(\frac{47}{28} a\right)^2} - \frac{\frac{4}{9} q}{\left(\frac{65}{86} a\right)^2} \right)$$

$$\vec{F}_+ = \frac{kq^2}{d^2}\left(\frac{4}{9}\left(\frac{28}{47}\right)^2 - \frac{4}{9}\left(\frac{86}{65}\right)^2\right)$$

$$= \frac{kq^2}{d^2}\left(0.2028 - 0.1363\right)$$

$$= 0.066\,\frac{kq^2}{d^2}$$

(2.3) $\sigma = \sigma_0 \left(\sin^2\phi - \frac{1}{2} \right)$

Trig identity $\sin^2\phi = \frac{1}{2} - \frac{1}{2}\cos 2\phi$

$$\sigma = -\frac{\sigma_0}{2}\cos 2\phi$$

Discard terms that blow up at ∞. The potential outside the cylinder is

$$V^o = \sum A_n s^{-n} \cos n\phi + B_n s^{-n} \sin n\phi \qquad + \cancel{\mathrm{Eo}}\ 5$$

$$\left.\frac{\partial V^o}{\partial s}\right|_a = \sum -n A_n a^{-(n+1)} \cos n\phi - n B_n a^{-(n+1)} \sin n\phi \qquad + 5$$

Electrostatic Boundary Condition

$$\left.\frac{\partial V^o}{\partial s}\right|_a - \underbrace{\left.\frac{\partial V^i}{\partial s}\right|_a}_{=0} = -\frac{\sigma}{\varepsilon_0} \qquad +5$$

$$\sigma = -\frac{\sigma_0}{2}\cos 2\phi = \varepsilon_0 \sum n\, a^{-(n+1)}\left[A_n \cos n\phi + B_n \sin n\phi\right]$$

By orthogonality, only $n=2$ is non-zero for A and all B are zero.

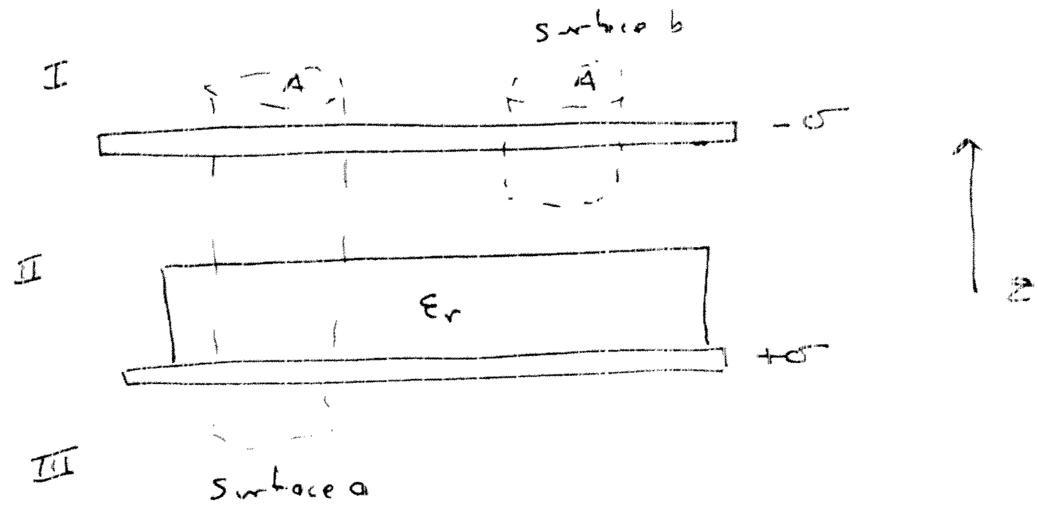
$$-\frac{\sigma_0}{2} = \varepsilon_0\, 2\, a^{-(2+1)} A_2$$

$$A_2 = -\frac{\sigma_0 a^3}{4\varepsilon_0} \qquad +5$$

The potential outside the cylinder is

$$V(s,\phi) = -\frac{\sigma_0 a^3}{4\varepsilon_0 s^2}\cos 2\phi \qquad +5$$

(2.4)



Surface a $\quad Q_{fenc} = 0$

$$\Phi_d = D_I A - D_{III} A = 0 \qquad \text{Gauss (Displacement)}$$

By symmetry, fields equal but opposite

$$D_I = -D_{III} \quad \Longrightarrow \quad \vec{D}_I = \vec{D}_{III} = 0$$

Surface b $\quad Q_{fenc} = -\sigma A$

$$D_I A - D_{II} A = Q_{fenc} = -\sigma A$$

$$\vec{D}_{II} = \sigma \hat{z} = \vec{D} \text{ inside dielectric}$$

Since dielectric linear,

$$\varepsilon_0 \varepsilon_r \vec{E}_r = \vec{D} = \sigma \hat{z}$$

$$\vec{E}_r = \frac{\sigma}{\varepsilon_0 \varepsilon_r} \hat{z}$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0(\varepsilon_r - 1)\vec{E}_{in}$$

$$= \varepsilon_0(\varepsilon_r - 1)\frac{\sigma}{\varepsilon_0 \varepsilon_r}\hat{z} = \frac{\varepsilon_r - 1}{\varepsilon_r}\sigma\hat{z}$$

Bound Charge Density

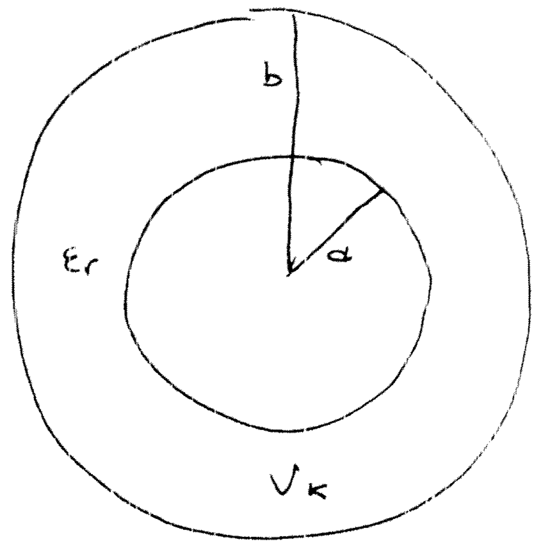
$$\rho_b = -\nabla \cdot \vec{P} = 0$$

<u>Surface Charge Densities</u>

Top surface $\sigma_t = \hat{z}\cdot\vec{P} = \frac{\varepsilon_r - 1}{\varepsilon_r}\sigma$

Bottom surface $\sigma_b = (-\hat{z})\cdot\vec{P} = \frac{-(\varepsilon_r - 1)}{\varepsilon_r}\sigma$

(2.5)



Solution to Laplaces Eqn keep terms that don't explode.

$$V_K = \sum_n \left(A_n r^n + B_n r^{-(n+1)}\right) P_n(\cos\theta)$$

$$V_0 = \sum_n C_n r^{-(n+1)} P_n(\cos\theta)$$

The applied potential is $V_0 P_1(\cos\theta)$ so only $n=1$ terms are needed.

$$V_K = \left[A_1 r + \frac{B_1}{r^2}\right] P_1(\cos\theta)$$

$$V_0 = \frac{C_1}{r^2} P_1(\cos\theta)$$

Boundary Conditions

$$V_k(a, \phi) = V_0 P_1(\cos\theta)$$

$$= \left(A_1 a + \frac{B_1}{a^2}\right) P_1(\cos\theta)$$

$$V_0 = A_1 a + \frac{B_1}{a^2}$$

Continuity at b

$$V_k(b, \phi) = V_o(b, \phi)$$

$$\left(A_1 b + \frac{B_1}{b^2}\right) P_1(\cos\theta) = \frac{C_1}{b^2} P_1(\cos\theta)$$

$$A_1 b + \frac{B_1}{b^2} = \frac{C_1}{b^2}$$

Electrostatic BC

$$\varepsilon_0 \left.\frac{\partial V_0}{\partial r}\right|_b - \varepsilon_r \varepsilon_0 \left.\frac{\partial V_k}{\partial r}\right|_b = -\sigma_f = 0$$

$$-2\frac{C_1}{b^3} - \varepsilon_r\left(A_1 - 2\frac{B_1}{b^3}\right) = 0$$

$$V_0 a^2 = A_1 a^3 + B_1 \qquad (1)$$

$$0 = A_1 b^3 + B_1 - C_1 \qquad (2)$$

$$0 = A_1 b^3 \varepsilon_r - 2\varepsilon_r B_1 + 2C_1 \qquad (3)$$

2(2) + (1)

$$A_1 b^3 (2+\varepsilon_r) + (2-2\varepsilon_r) B_1 = 0$$

$$B_1 = -\frac{A_1 b^3}{2}\,\frac{2+\varepsilon_r}{1-\varepsilon_r} = \frac{A_1 b^3}{2}\,\frac{2+\varepsilon_r}{\varepsilon_r - 1} \qquad (4)$$

(4) → (1)

$$V_0 a^2 = A_1 a^3 + \frac{A_1 b^3}{2}\,\frac{2+\varepsilon_r}{\varepsilon_r - 1}$$

$$= A_1\left(a^3 + \frac{b^3}{2}\,\frac{2+\varepsilon_r}{\varepsilon_r - 1}\right)$$

$$A_1 = \frac{V_0 a^2}{a^3 + \frac{b^3}{2}\,\frac{2+\varepsilon_r}{\varepsilon_r - 1}} \qquad (5)$$

(5) $\rightarrow$ (4)

$$B_1 = \frac{A_1 b^3}{2} \frac{2+\varepsilon_r}{\varepsilon_r - 1}$$

$$= \frac{V_0 a^2}{a^3 + \frac{b^3}{2}\left(\frac{2+\varepsilon_r}{\varepsilon_r-1}\right)} \cdot \frac{b^3}{2} \frac{2+\varepsilon_r}{\varepsilon_r - 1}$$

$$C_1 = A_1 b^3 + B_1$$

$$= \frac{V_0 a^2 b^3}{a^3 + \frac{b^3}{2}\left(\frac{2+\varepsilon_r}{\varepsilon_r-1}\right)} + \frac{V_0 a^2 b^3}{a^3 + \frac{b^3}{2}\left(\frac{2+\varepsilon_r}{\varepsilon_r-1}\right)} \cdot \frac{2+\varepsilon_r}{2(\varepsilon_r - 1)}$$

(2.6) $\vec{P} = \gamma r^2 \hat{r}$

$$\rho_b = -\nabla \cdot \vec{P} = -\gamma \frac{1}{r^2} \frac{\partial}{\partial r} r^4$$

$$= -4\gamma r$$

Surface charge $\sigma_b = \vec{P} \cdot \hat{r} = \gamma a^2$

<u>Inside Object</u> $r < a$ Gaussian surface radius r

$$Q_{enc} = \int \rho \, d\tau$$

$$= \int \rho \, r^2 \sin\theta \, dr \, d\phi \, d\theta$$

$$= 4\pi \int_0^r \rho \, r^2 \, dr$$

$$= -16\gamma\pi \int_0^r r^3 \, dr$$

$$= -4\gamma\pi r^4$$

<u>Gauss Law</u>

$$\Phi_e = \int \vec{E} \cdot \hat{n} \, dA = 4\pi r^2 E = \frac{Q_{enc}}{\epsilon_0}$$

$$\vec{E}_i = \frac{Q_{enc}}{4\pi\epsilon_0 r^2} \hat{r} = \frac{-4\gamma\pi r^4}{4\pi\epsilon_0 r^2} \hat{r}$$

$$= -\frac{\gamma r^2}{\epsilon_0} \hat{r}$$

Field Outside   Charge enclosed is total charge of volume charge plus surface charge.

$$Q_{enc} = Q_{vol} + Q_{surface}$$

$$= -4\gamma\pi a^4 \quad + (\gamma a^2)(4\pi a^2)$$

$$= 0$$

$$\vec{E}_o = 0$$

An alternate, but cool, solution I didn't think of

$Q_{f\,enc} = 0$ everywhere

by symmetry $\Longrightarrow$ $\vec{D} = 0$ everywhere.

<u>Outside</u> $\vec{D} = \varepsilon_0 \vec{E}_o \Longrightarrow \vec{E}_o = 0$

<u>Inside</u> $\vec{D} = \varepsilon_0 \vec{E}_i + \vec{P} = 0$

$$\vec{E}_i = -\frac{\vec{P}}{\varepsilon_0} = \frac{-\gamma r^2}{\varepsilon_0}\hat{r}$$