Practice Test 3 - Spring 2013

Work four of the six problems. Place the problems in the order you wish them graded. The first two problems form the first half test; the second two problems form the second half test.

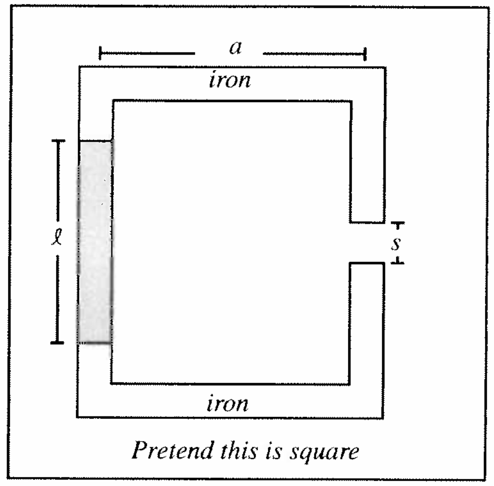
Problem 3.1 Two straight wires carry a current I in the $+\hat{y}$ direction. The wires run through the points $\pm\ell\hat{x}$. The wires have length D and are centered on the x axis. Compute the vector potential at the origin.

Problem 3.2 A cylindrical conductor of radius a has a current density that increases with radius, $\vec{J}(s) = J_0 s^2 \hat{z}$. The current density is zero outside the wire. Compute the magnetic field everywhere.

Problem 3.3 As part of an honors project in UPII this semester, a student used a stack of NdFeB magnets to power a hand generator. Each magnet was a cylinder of height $h = 1\text{mm}$ and radius $r = 1\text{cm}$. The student made the approximation that the magnetic field of a stack of six magnets was six times the magnetic field of one. The magnetization of NdFeB is $1.02 \cdot 10^6 \text{A/m}$. Compare the field at the center of a single magnet with the field at the center of a stack of six magnets. You may model the bound current of a single magnet as a ring, but may not use this model for the stack. You may use the formula for the field of the current distribution of the stack if you included it on your formula card, if not you will have to re-derive it.

Problem 3.4 A thin square wafer has constant magnetization density $\vec{M} = M_0\hat{z}$. The wafer is ℓ long on each side and has thickness d in the $\hat{z}$ direction. Compute the magnetic field at the center of the wafer.

Problem 3.5 A horseshoe magnet is formed into a square. It is made using a section of NdFeB permanent magnetic material with magnetization density $1.02 \cdot 10^6 \text{A/m}$ of length $\ell = 2\text{cm}$. The square has average side length $a = 3\text{cm}$ and a gap with width $s = 1\text{cm}$. Compute the magnetic field in the gap. The relative permeability of iron is 100. Note, this problem may be worked in the same manner as the other magnets formed of rings of various materials worked in the homework.

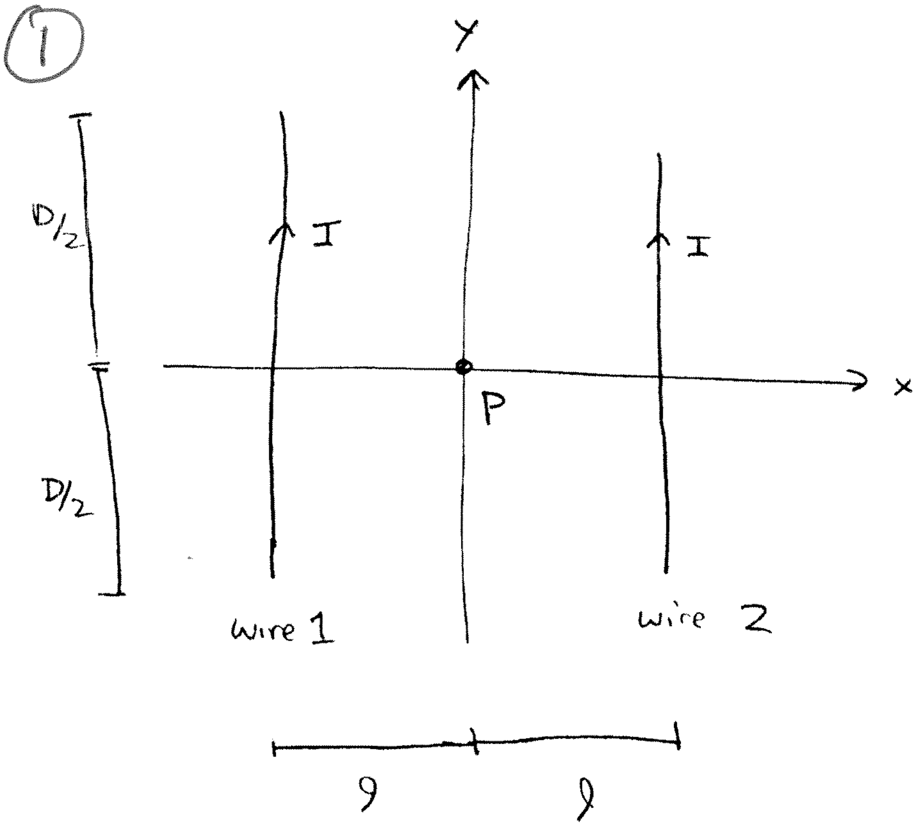


Problem 3.6 A cylindrical stainless steel channel with relative permeability μ_r, inner radius a and outer radius b, confines a plasma with current density

$$\vec{J} = \frac{J_0 a}{s} \sin\left(\frac{\pi s}{a}\right) \hat{z}$$

where the axis of the solenoid is the z-axis and J_0 is a constant. Compute $\vec{H}$, $\vec{B}$, and $\vec{M}$ everywhere.

(1)



The vector potential is given by

$$\vec{A}_1 = \frac{\mu_0}{4\pi}\int \frac{d\vec{I}_1}{r}$$

$$= \frac{\mu_0 I \hat{y}}{4\pi}\int_{-D/2}^{D/2} \frac{dy}{\sqrt{l^2+y^2}}$$

Evidently $\vec{A}_1 = A_2$

Perform the integral,

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right) \quad \text{Schaum}$$

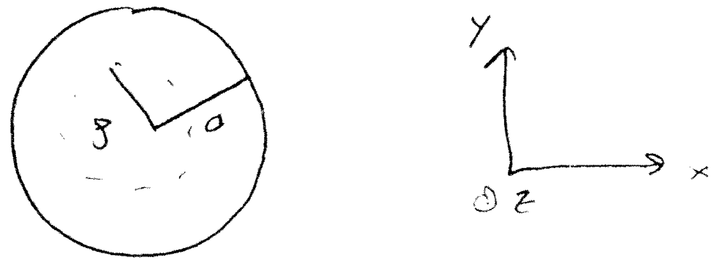
$$\vec{A}_1 = \frac{\mu_0 I \hat{y}}{4\pi} \ln\left(y + \sqrt{y^2+\ell^2}\right)\Big|_{-D/2}^{D/2}$$

$$= \frac{2\mu_0 I \hat{y}}{4\pi} \ln\left(y + \sqrt{y^2+\ell^2}\right)\Big|_{0}^{D/2}$$

$$= \frac{\mu_0 I \hat{y}}{2\pi} \ln\left(\frac{D/2 + \sqrt{(D/2)^2 + \ell^2}}{\ell}\right)$$

$$\vec{A} = \vec{A}_1 + \vec{A}_2 = \frac{\mu_0 I \hat{y}}{\pi} \ln\left(\frac{D}{2\ell} + \sqrt{1 + \left(\frac{D}{2\ell}\right)^2}\right)$$

2 For this solution $\rho \to s$. Old notation.



The current flowing through a surface bounded by a circular Amperian path of radius ρ is

$$I_{enc} = \int_0^{\rho} 2\pi \rho \, d\rho \, J$$

$$= 2\pi J_0 \int_0^{\rho} \rho^3 \, d\rho$$

$$= \frac{\pi J_0}{2} \rho^4$$

If $\rho < a$ and

$$I_{enc} = \frac{\pi J_0}{2} a^4$$

if $\rho > a$.

The magnetic field is found from Ampere's Law,

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$

$$2\pi\rho B = \mu_0 I_{enc}$$

$$B = \frac{\mu_0 I_{enc}}{2\pi\rho}$$

For $\rho < a$,

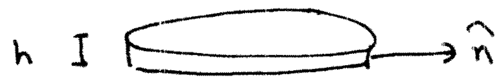
$$\vec{B} = \frac{\mu_0}{2\pi\rho}\left(\frac{\pi J_0}{2}\rho^4\right)\hat{\phi} = \frac{\mu_0 J_0 \rho^3}{4}\hat{\phi}$$

For $\rho > a$,

$$\vec{B} = \frac{\mu_0}{2\pi\rho}\left(\frac{\pi J_0 a^4}{2}\right) = \frac{\mu_0 J_0 a^4}{4\rho}\hat{\phi}$$

Direction found by Right Hand Rule.

3.3



The bound surface current is

$$\vec{K}_b = \vec{M} \times \hat{s} = M_0 \hat{\phi}$$

For one magnet, treat it as a ring of current with current $I = K_b h = M_0 h$

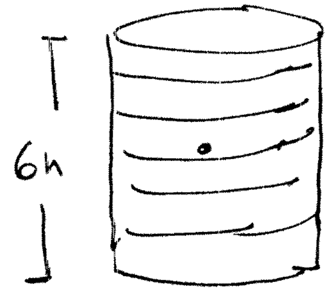
The field of a ring at the origin is

$$B_r = \frac{\mu_0 I}{2r} = \frac{\mu_0 M_0 h}{2r}$$

$$= \frac{\left(4\pi \times 10^{-7} \frac{Tm}{A}\right)\left(1.02 \times 10^{6} \frac{A}{m}\right)(0.001m)}{2(0.01m)}$$

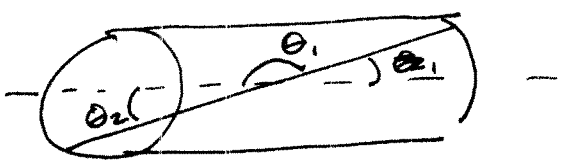
$$= 0.064\,T$$

The stack of 6 magnets must be modeled as a finite solenoid



$K_b = M_0 = nI$

The field of a finite solenoid



$$B_s = \frac{\mu_0 n I}{2}\left(\cos\theta_2 - \cos\theta_1\right)$$

$$= \frac{\mu_0 n I}{2}\left(\cos\theta_2 + \cos\theta_2\right)$$

$$= \mu_0 n I \cos\theta_2$$

$$\cos\theta_2 = \frac{3h}{\sqrt{(3h)^2 + r^2}} = \frac{3(0.001m)}{\sqrt{(0.003m)^2 + (0.01m)^2}}$$

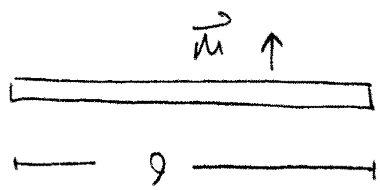
$$= 0.287$$

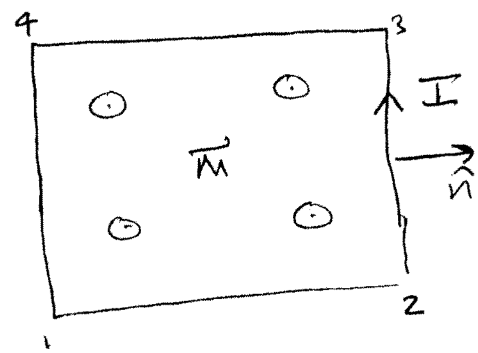
$$B_s = \mu_0 K_b \cdot \cos\theta_2 = \mu_0 M \cdot 0.287 = 0.37T$$

$$= 6 \cdot (0.061T)$$

Not much of a correction.

④





The magnetization produces a surface current density

$$\vec{K}_b = \vec{M} \times \hat{n} = M_0 \hat{z} \times \hat{n}$$

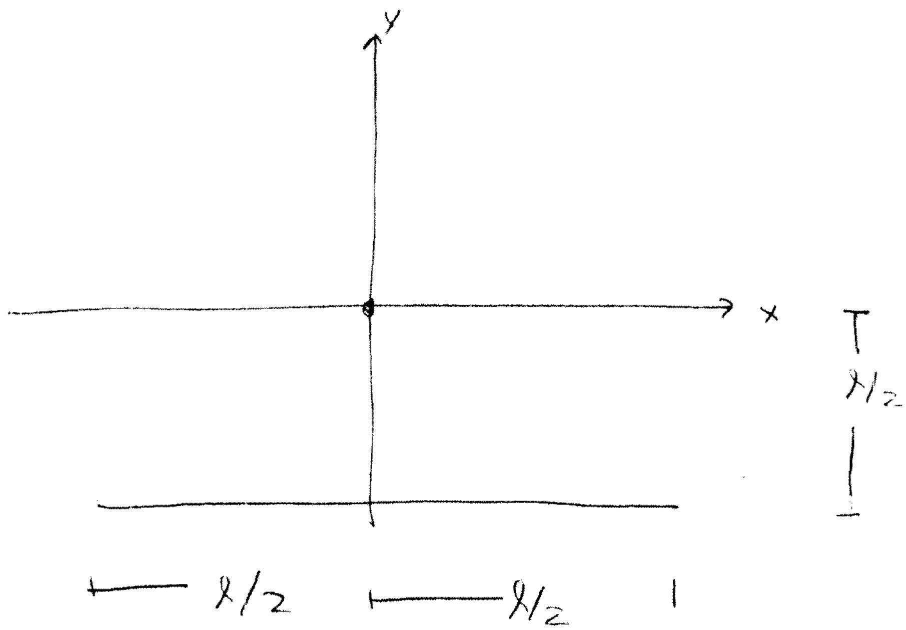
$$|\vec{K}_b| = M_0$$

Since the wafer is thin, this produce an effective current $I = |K_b| d = M_0 d$ around the outer edge.

The field of each edge is evidently the same, so the total field at the center is

$$\vec{B}_0 = 4 \vec{B}_{12}$$

The field from segment 1,2 is



The displacement vect. from $\vec{r}_i = (x, -\ell/2, 0)$

to $\vec{r}_c = (0,0,0)$ is

$$\vec{r}\,'' = (-x, \ell/2, 0)$$

The current is $\vec{I} = I\,d\vec{\ell} = I\,dx\,\hat{x}$

and the required cross-product

$$\vec{I} \times \vec{r}\,'' = \frac{I\,dx\,\ell}{2}\,\hat{z}$$

Biot - Savart

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{d\vec{I} \times \vec{r}''}{r''^3}$$

$$= \frac{\mu_0}{4\pi} \int_{-l/2}^{l/2} \frac{I l \, dx \, \hat{z}}{2(x^2 + (l/2)^2)^{3/2}}$$

$$= 2 \cdot \frac{\mu_0 I l \hat{z}}{8\pi} \int_0^{l/2} \frac{dx}{(x^2 + (l/2)^2)^{3/2}}$$

$$= \frac{\mu_0 I l}{4\pi} \hat{z} \left[\frac{x}{(l/2)^2 \sqrt{x^2 + (l/2)^2}} \right]_0^{l/2}$$

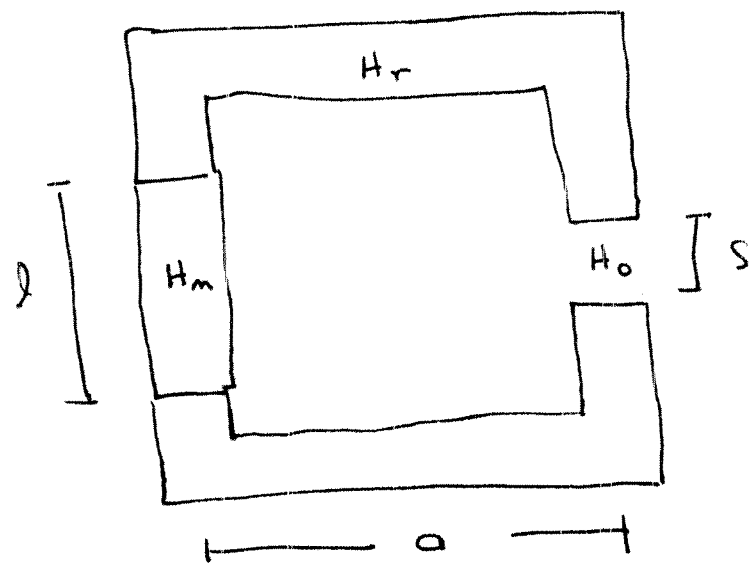
$$= \frac{\mu_0 I l}{4\pi} \hat{z} \left(\frac{l/2}{\sqrt{2}\,(l/2)^3} \right)$$

$$= \frac{\mu_0 I}{2\pi\sqrt{2}} \cdot \frac{1}{l/2} \hat{z} = \frac{\mu_0 I}{\sqrt{2}\,\pi l} \hat{z}$$

The total field is $4\vec{B}$

$$\vec{B}_0 = \frac{4\mu_0 I}{\sqrt{2}\,\pi l} \hat{z}$$

(5)



<u>Ampere's Law</u>

$$\oint \vec{H} \cdot d\vec{l} = I_{fenc} = 0$$

$$H_m l + H_o s + H_r(4a - l - s) = 0$$

Using <u>No Magnetic Monopoles</u>, the magnetic field must be the same at all points around the circle.

$$B = B_o = B_i = B_m$$

<u>Outside</u> $\mu_o H_o = B_o = B$

<u>In the iron</u> $B_r = B = \mu_r \mu_o H_r$

In the magnetic material,

$$H_m = \frac{B}{\mu_0} - M$$

Substitute,

$$\left(\frac{B}{\mu_0} - M\right) l + \frac{B}{\mu_0} s + \frac{B}{\mu_0 \mu_r} \overbrace{\left(4a - l - s\right)}^{\gamma} = 0$$

$$Bl - \mu_0 M l + Bs + \frac{B}{\mu_r}\gamma = 0$$

$$B\left(l + s + \frac{\gamma}{\mu_r}\right) = \mu_0 M l$$

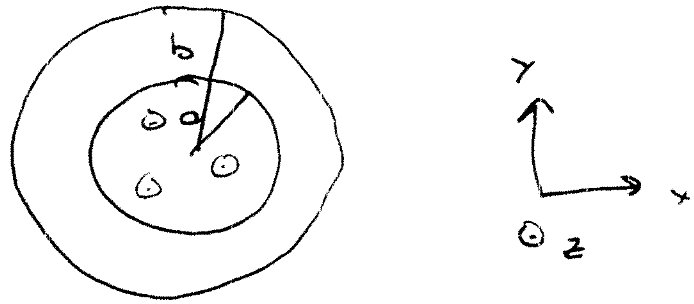
$$B = \frac{\mu_0 M l}{l + s + \frac{\gamma}{\mu_r}}$$

$$\gamma = (4 \cdot 3cm - 2cm - 1cm)$$
$$= 9cm$$

$$B = \frac{(4\pi \times 10^{-7} Tm/A)(1.02 \times 10^6 A/m)(0.02m)}{\left(0.02m + 0.01m + \frac{0.09m}{100}\right)}$$

$$B = \cancel{0.66T} \; 0.83T$$

(6)



<u>Compute $\vec{H}$</u>

<u>$s < a$</u>

$$I_{fenc} = \int_0^{2\pi} d\phi \int_0^{s} s\, J\, ds$$

$$= 2\pi \cdot J_0 a \cdot \int_0^{s} \sin\frac{\pi s}{a}\, ds$$

$$= -2\pi J_0 a \cdot \frac{a}{\pi} \cos\frac{\pi s}{a} \Big|_0^{s}$$

$$= 2 J_0 a^2 \left(1 - \cos\frac{\pi s}{a}\right)$$

<u>Ampere's Law</u>

$$\int \vec{H} \cdot d\vec{l} = I_{fenc} = 2\pi s\, H$$

RHR

$$\vec{H} = \frac{I_{fenc}}{2\pi s}\,\hat{\phi} = \frac{2 J_0 a^2}{2\pi s}\left(1 - \cos\frac{\pi s}{a}\right)\hat{\phi}$$

Since there is no material, $\vec{M} = 0$ $s < a$, and

$$\vec{B} = \mu_0 \vec{H} = \frac{J_0 a^2 \mu_0}{\pi s}\left(1 - \cos\frac{\pi s}{a}\right)$$

For $s > a$

$$I_{fenc} = 2 J_0 a^2 \left(1 - \cos\frac{\pi a}{a}\right)$$

$$= 4 J_0 a^2$$

$$\vec{H} = \frac{I_{fenc}}{2\pi s}\hat{\phi} = \frac{J_0 a^2}{\pi s}\hat{\phi}$$

For $s > b$, $\vec{M} = 0$, $\vec{B} = \mu_0 \vec{H} = \frac{J_0 a^2 \mu_0}{\pi s}\hat{\phi}$

For $a < s < b$, $\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H}$

$$\vec{B} = \frac{\mu_r \mu_0 J_0 a^2}{\pi s}\hat{\phi}$$

$$\vec{M} = \chi_m \vec{H} = (\mu_r - 1)\vec{H}$$

$$= \frac{2(\mu_r - 1) J_0 a^2}{\pi s}\hat{\phi}$$