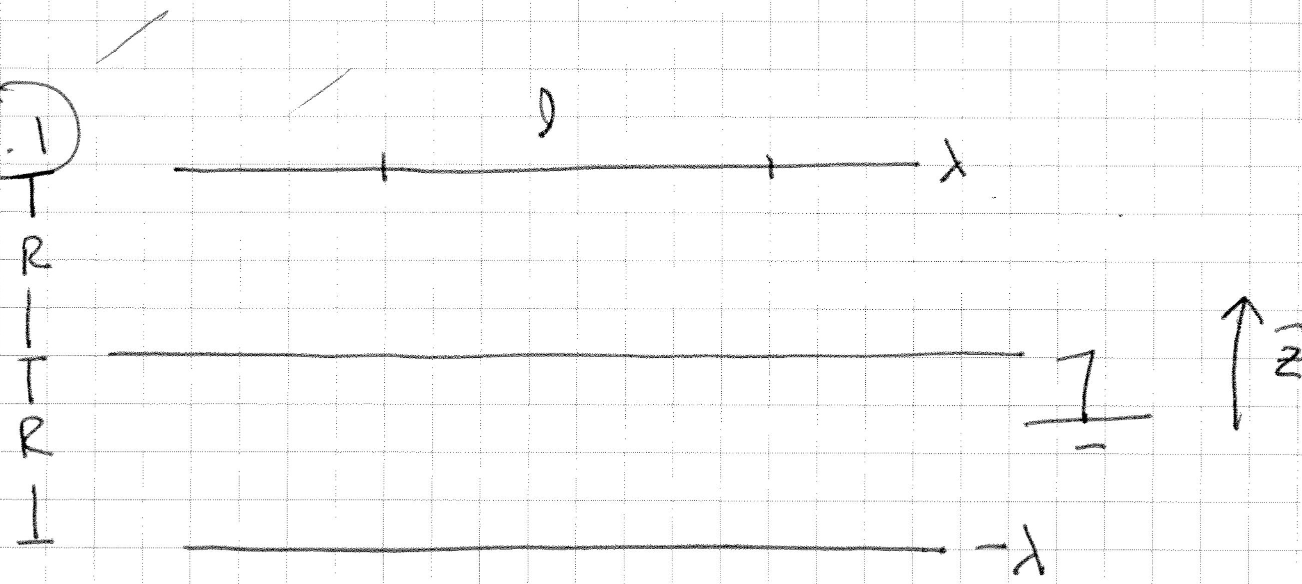
2.1



Place image linear charge a distance R below the plane. The field at the top linear charge is

$$\vec{E} = \frac{-\lambda}{2\pi\varepsilon_0 (2R)} \hat{z}$$

The force this field exerts on a length l of the top linear charge is

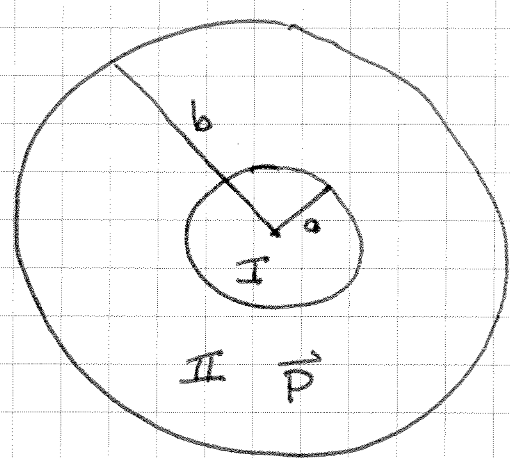
$$\vec{F} = q\vec{E} \qquad q = \lambda l$$

$$= \lambda l \vec{E}$$

$$= \frac{-\lambda^2 l}{4\pi\varepsilon_0 R} \hat{z}$$

$$\vec{F}/l = \frac{-\lambda^2}{4\pi\varepsilon_0 R} \hat{z}$$

(2.2)



The polarization produces a bound charge density $\sigma_{bi} = -\hat{s}\cdot\vec{P} = -P_0$ on the inner surface and $\sigma_{bo} = \hat{s}\cdot\vec{P} = +P_0$ on the outer surface. It also produces a volume charge density

$$\rho_b = -\nabla\cdot P = -\frac{1}{s}\frac{\partial}{\partial s} s P_0$$

$$= -\frac{P_0}{s}$$

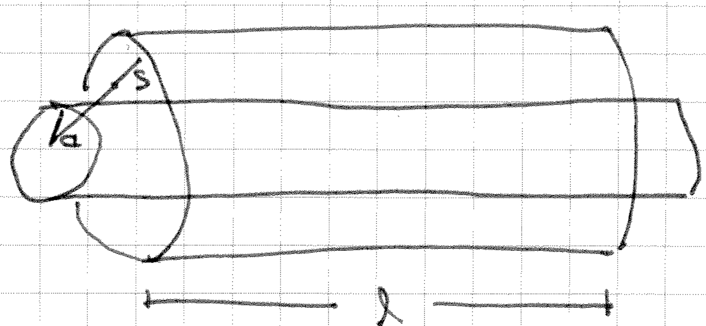
in the interior of the dielectric.

Region I $(s<a)$ $\quad Q_{enc} = 0$

$$\Rightarrow \vec{E}_I = 0$$

$$\Rightarrow \Delta V_{0a} = 0$$

Region II   Use a cylindrical Gaussian surface of length l and radius s



$$Q_{enc} = 2\pi a l \sigma_{bi} + l \int_a^s \rho_b \, ds \, s \, d\phi$$

$$= -2\pi a l P_0 - P_0 l \int_a^s ds \, d\phi$$

$$= -2\pi a l P_0 - P_0 l \, 2\pi (s - a)$$

$$= -P_0 l \, 2\pi s$$

Gauss Law

$$\Phi_e = 2\pi s l E = \frac{Q_{enc}}{\varepsilon_0} = \frac{-P_0 l 2\pi s}{\varepsilon_0}$$

$$\vec{E}_{II} = \frac{-P_0}{\varepsilon_0} \hat{s}$$

The potential difference is then

$$\Delta V_{ab} = \Delta V_{ab} = -\int_a^b \vec{E}_{II} \cdot d\vec{\ell}$$

$$= -\int_a^b \frac{-P_0}{\varepsilon_0}\, ds$$

$$= +\frac{P_0}{\varepsilon_0}(b-a)$$

Alternate Solution

The system is ~~sphe~~ cylindrically symmetric and there is no free charge anywhere, so

$$\vec{D}_I = \vec{D}_{II} = 0$$

In region I, $\vec{P}_I = 0 \Rightarrow \vec{E}_I = 0$

In region II, $\vec{P}_{II} = P_0 \hat{s}$, $\vec{D}_{II} = \varepsilon_0 \vec{E}_{II} + \vec{P}_{II} = 0$

$$\vec{E}_{II} = -\frac{\vec{P}_{II}}{\varepsilon_0} = \frac{-P_0}{\varepsilon_0}\hat{s}$$

(2.3)

$$\sigma(a,\theta) = \sigma_0 P_0 + \sigma_0 P_1(\cos\theta)$$

Spherical

Inside $r<a$

$$V_i(s,\theta) = \sum_n A_n r^n P_n(\cos\theta)$$

Outside $r>a$

$$V(s,\theta) = \sum_n B_n r^{-(n+1)} P_n(\cos\theta)$$

Because the boundary condition only involves P_0, P_1 these will be the only harmonics in the solution

$$V_i(s,\theta) = A_0 P_0 + A_1 r^1 P_1(\cos\theta)$$

$$V_o(s,\theta) = \frac{B_0}{r} + \frac{B_1}{r^2} P_1(\cos\theta)$$

Apply boundary conditions

Continuity $\quad V_i(a,\theta) = V_o(a,\theta)$

$$A_0 P_0 + A_1 a P_1 = \frac{B_0}{a} + \frac{B_1}{a^2} P_1$$

Equate like terms in P_i

$$A_0 = \frac{B_0}{a}$$

$$A_1 a = \frac{B_1}{a^2}$$

$$B_0 = a A_0$$

$$B_1 = a^3 A_1$$

<u>Apply Electrostatic Boundary Condition</u>

$$\left.\frac{\partial V_o}{\partial r}\right|_a - \left.\frac{\partial V_i}{\partial r}\right|_a = -\frac{\sigma}{\varepsilon_0}$$

$$\frac{\partial V_o}{\partial r} = -\frac{B_0}{r^2} P_0 - \frac{2 B_1}{r^3} P_1$$

$$\frac{\partial V_i}{\partial r} = A_1 P_1$$

$$\left.\frac{\partial V_o}{\partial r}\right|_a - \left.\frac{\partial V_i}{\partial r}\right|_a = \frac{-\sigma}{\varepsilon_0}$$

$$-\frac{B_0}{a^2} - \frac{2 B_1 P_1}{a^3} - A_1 P_1 = \frac{-\sigma_0 - \sigma_0 P_1}{\varepsilon_0}$$

Equate like terms in P

P_0:
$$-\frac{B_0}{a^2} = -\frac{\sigma_0}{\varepsilon_0}$$

$$B_0 = a^2 \frac{\sigma_0}{\varepsilon_0}$$

$$A_0 = a \frac{\sigma_0}{\varepsilon_0}$$

P_1:
$$-\frac{2B_1}{a^3} - A_1 = -\frac{\sigma_0}{\varepsilon_0}$$

$$B_1 = a^3 A_1$$

$$-2A_1 - A_1 = -\frac{\sigma_0}{\varepsilon_0}$$

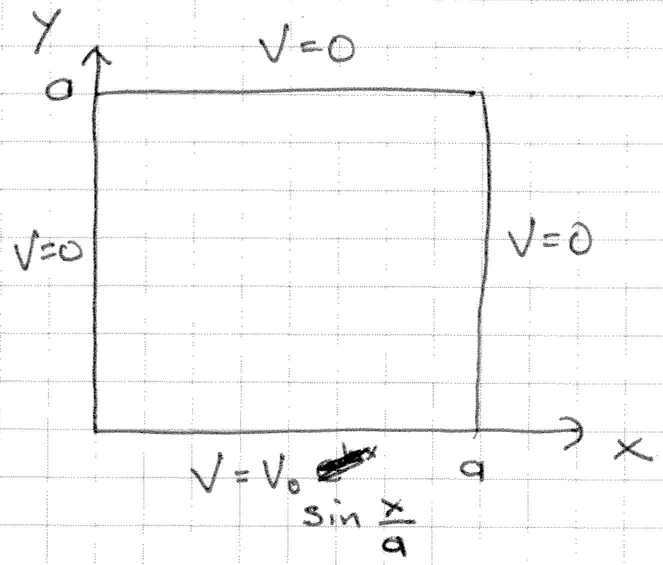
$$A_1 = \frac{\sigma_0}{3\varepsilon_0}$$

$$B_1 = \frac{a^3\sigma_0}{3\varepsilon_0}$$

$$V_i = \frac{a\sigma_0}{\varepsilon_0} + \frac{\sigma_0}{3\varepsilon_0} r\cos\theta$$

$$V_o = \frac{\sigma_0 a^2}{\varepsilon_0 r} + \frac{a^3\sigma_0}{3\varepsilon_0 r^2}\cos\theta$$

(2.4)



Laplace's Eqn

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V = X\,Y$$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{= -k^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{k^2} = 0$$

$$\frac{1}{Y}\frac{d^2 Y}{dy^2} = +k^2 \qquad \frac{1}{X}\frac{d^2 X}{dx^2} = -k^2$$

$$\frac{d^2 Y}{dy^2} - k^2 Y = 0 \qquad \frac{d^2 X}{dx^2} + k^2 X = 0$$

$$Y = e^{\pm ky} \qquad X = \cos kx,\ \sin kx$$

Work on $X(x)$

$$X(0, y, z) = 0 \quad \Rightarrow \quad X = A \sin kx$$

$$X(a, y, z) = 0 \quad \Rightarrow \quad k = \frac{n\pi}{a}$$

$$X(x) = \sin k_n x \qquad k_n = \frac{n\pi}{a}$$

General Solution

$$V(x, y) = \sum_n \left(A_n e^{k_n y} + B_n e^{-k_n y}\right) \sin k_n x$$

Boundary Condition

$$V(x, a) = 0 = \sum \left(A_n e^{k_n a} + B_n e^{-k_n a}\right) \sin k_n x$$

Use orthogonality to assert coefficients of each power of sine must be zero

$$A_n e^{k_n a} + B_n e^{-k_n a} = 0$$

$$B_n = -A_n e^{2k_n a}$$

Boundary Condition

$$V(x,0) = V_0 \sin\frac{\pi x}{a} = \sum_n (A_n + B_n) \sin k_n x$$

Again, associate like sines

If $n \neq 1$, $A_n + B_n = 0 \Rightarrow A_n = B_n = 0$

If $n = 1$,

$$V_0 = A_n + B_n = A_1 + B_1$$

$$= A_1 + \left(-A_1 e^{2k_1 a}\right)$$

$$= A_1 \left(1 - e^{2k_1 a}\right)$$

$$A_1 = \frac{V_0}{1 - e^{2k_1 a}}$$

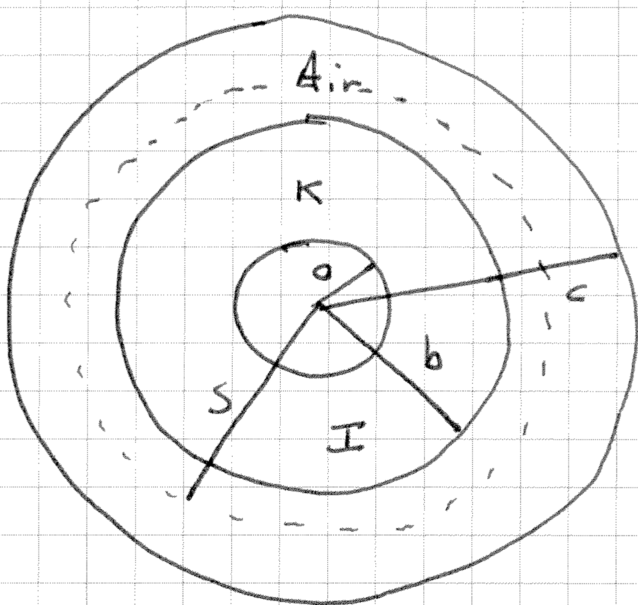
$$B_1 = -\frac{V_0 e^{2k_1 a}}{1 - e^{2k_1 a}}$$

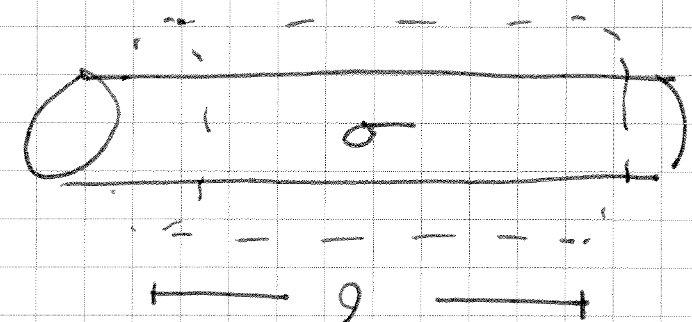
$$V(x,y) = \cancel{V_0 e^{k_1 x}}$$

$$\left[\frac{V_0 e^{k_1 y}}{1 - e^{2k_1 a}} - \frac{V_0 e^{2k_1 a} e^{-k_1 y}}{\left(1 - e^{2k_1 a}\right)}\right] \sin k_1 x$$

$$k_1 = \frac{\pi}{a}$$

2.5





Use a cylindrical Gaussian surface of radius s coaxial with the cylinder.

In region I and II $a < s < c$, the free charge enclosed is $Q_{f\,enc} = 2\pi a l \sigma$

The displacement flux is

$$\Phi_d = 2\pi s l D = Q_{f\,enc} = 2\pi a l \sigma$$

$$\vec{D}_{II} = \vec{D}_{I} = \frac{a\sigma}{s}\,\hat{s}$$

The electric field in Region II $\quad b < s < c$

is $\quad \vec{D}_{II} = \varepsilon_0 \vec{E}_{II}$

$$\vec{E}_{II} = \frac{\vec{D}_{II}}{\varepsilon_0} = \frac{a\sigma}{s\varepsilon_0}\hat{s}$$

and the potential difference

$$\Delta V_{bc} = -\int_b^c \vec{E}_{II} \cdot d\vec{l} \qquad d\vec{l} = ds\,\hat{s}$$

$$= -\int_b^c \frac{a\sigma\, ds}{s\varepsilon_0}$$

$$= -\frac{a\sigma}{\varepsilon_0} \ln\left(c/b\right)$$

The electric field in Region I $\quad a < s < b \quad$ is

$$\vec{D}_I = \kappa \varepsilon_0 \vec{E}_I$$

$$\vec{E}_I = \frac{a\sigma}{s\kappa\varepsilon_0}\hat{s}$$

The potential difference is

$$\Delta V_{ab} = -\int_a^b \vec{E}_I \cdot d\vec{l} = -\int_a^b \frac{a\sigma}{s\kappa\varepsilon_0} ds$$

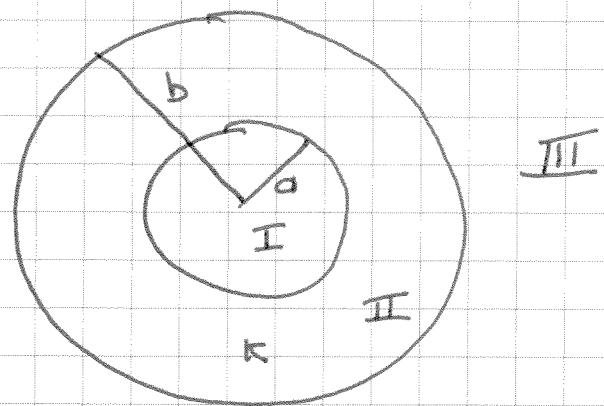
$$= -\frac{a\sigma}{\kappa\varepsilon_0}\ln\left(b/a\right)$$

The total potential difference is

$$\Delta V_{ac} = \Delta V_{ab} + \Delta V_{bc}$$

$$= -\frac{a\sigma}{\kappa\varepsilon_0} \ln\left(b/a\right) - \frac{a\sigma}{\varepsilon_0} \ln\left(\frac{b}{c}\right)$$

2.6



Displacement Field

Region I $\quad r < a$

$$Q_{fenc} = \frac{4}{3}\pi \rho r^3$$

$$\Phi_d = 4\pi r^2 D_I = Q_{fenc}$$

$$\vec{D}_I = \frac{\rho r}{3}\hat{r}$$

Region II + III

$$Q_{fenc} = \frac{4}{3}\pi a^3 \rho$$

$$\vec{D}_{II} = \vec{D}_{III} = \frac{\frac{4}{3}\pi a^3 \rho}{4\pi r^2}\hat{r}$$

$$= \frac{a^3 \rho}{3 r^2}\hat{r}$$

Electric Field + Polarization

Region I $\qquad \vec{P}_I = 0$

$$\vec{D}_I = \varepsilon_0 \vec{E}_I$$

$$\vec{E}_I = \frac{\rho r}{3\varepsilon_0}\hat{r}$$

Region II $\qquad$ Linear Dielectric

$$\vec{D}_{II} = \kappa \varepsilon_0 \vec{E}_{II}$$

$$\vec{E}_{III} = \frac{a^3 \rho}{3\varepsilon_0 r^2 \kappa}\hat{r}$$

$$\vec{P}_{II} = \chi_e \varepsilon_0 \vec{E}$$

$$= \frac{(\kappa-1)a^3\rho}{3\kappa r^2}\hat{r}$$

Region II $\qquad \vec{P}_{III} = 0$

$$\vec{E}_{III} = \vec{D}_{III}/\varepsilon_0 = \frac{a^3\rho}{3\varepsilon_0 r^2}\hat{r}$$

or using $k_1 = \pi/a$

$$V(x,y) = V_0\left(\frac{e^{\pi y/a}}{1-e^{2\pi}} - \frac{e^{2\pi}e^{-\pi y/a}}{1-e^{2\pi}}\right)\sin\frac{\pi x}{a}$$

$$= \frac{V_0}{1-e^{2\pi}}\left(e^{\pi y/a} - e^{2\pi-\pi y/a}\right)\sin\frac{\pi x}{a}$$

$$= \frac{V_0\,e^{\pi}}{1-e^{2\pi}}\left(e^{\pi+\pi y/a} - e^{\pi-\pi y/a}\right)\sin\frac{\pi x}{a}$$

$$= \frac{V_0}{e^{-\pi}-e^{\pi}}\left(e^{\pi+\pi y/a} - e^{\pi-\pi y/a}\right)\sin\frac{\pi x}{a}$$