Vector Operators

Del (nabla) scalar $\rightarrow$ vector

$$\nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

Gradient (grad) - scalar $\rightarrow$ vector

$$\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$\Rightarrow$ Vector that points in the direction of the greatest rate of change of f.

Divergence vector $\rightarrow$ scalar

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$\Rightarrow$ The rate at which $\vec{A}$ spreads out.

Curl - vector → vector

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$\Rightarrow$ Rotation of $\vec{A}$

Laplacian scalar → scalar

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial f^2}{\partial z^2}$$

Properties

- Divergence and curl must be applied to a vector; $\nabla \times f$ and $\nabla \cdot f$ are nonsense.
- We can define the Laplacian of a vector as

$$\nabla^2 \vec{A} = \left(\nabla^2 A_x , \nabla^2 A_y , \nabla^2 A_z \right)$$

Working with operators

- ∇, $\nabla\cdot$, $\nabla\times$, ∇^2 are operators.

- Operators, in general, do not commute.

  $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ does not imply

  $\vec{r}\cdot\nabla = \nabla\cdot\vec{r}$

  $$(\vec{r}\cdot\nabla)f = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)f$$

  $$(\nabla\cdot\vec{r})f = \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)f$$

  $$= 3f$$

  * Note that location of parenthesis is important.

  * Note that to prove something about an operator, you have to let is operate on something.

- In general, to show two operators are equal, $O_1 = O_2$, we show that $O_1 f = O_2 f$ for all f.

Operator Identities - The rules of one-dimensional calculus can be extended to vector operators.

Product Rule

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$\Longrightarrow$ three vector products

$$\underline{\nabla(fg)} = f\nabla g + g\nabla f$$

$$\underline{\nabla\cdot(\vec{A}\times\vec{B})} = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

$\Longrightarrow$ Note - cannot be derived by applying $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A})$ because operators do not commute.

$$\underline{\nabla\times(\vec{A}\times\vec{B})} = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$$

What does that last one mean?

1) $(\vec{B}\cdot\nabla)\vec{A} \neq \vec{A}(\vec{B}\cdot\nabla)$

$$= \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z} \right) \vec{A}$$

$$= \left(B_x \frac{\partial A_x}{\partial x} + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z} \right) \hat{x}$$

$$+ \left(B_x \frac{\partial A_y}{\partial x} + B_y \frac{\partial A_y}{\partial y} + B_z \frac{\partial A_y}{\partial z} \right) \hat{y}$$

$$+ \left(B_x \frac{\partial A_z}{\partial x} + B_y \frac{\partial A_z}{\partial y} + B_z \frac{\partial A_z}{\partial z} \right) \hat{z}$$

$\Longrightarrow$ Vector notation can be very compressed

2) $\vec{A}(\nabla\cdot\vec{B}) = (\nabla\cdot\vec{B})\vec{A}$

$$= \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) A_x \hat{x}$$

$$+ \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) A_y \hat{y}$$

$$+ \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) A_z \hat{z}$$

Second Dervatives

$$\nabla \times \nabla f = 0$$

$$(\nabla \times \nabla) \cdot \vec{A} = 0$$

$$(\nabla \times \nabla) \times \vec{A} = 0$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$