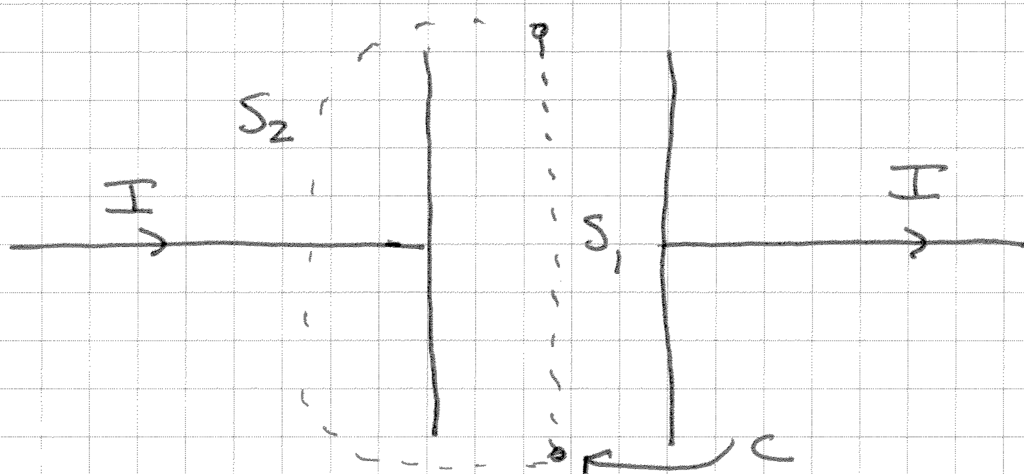
Ampere's Law for Electrodynamics

Ampere's Law

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

is incomplete.

Consider charging a capacitor with current I and two ~~curves C_1 and C_2~~.



surfaces S_1 and S_2 bounded by the same curve C.

The current encircled for S_1, $I_{enc,1} = 0$ but the current encircled for S_2, $I_{enc,2} = I$.

This means $\oint \vec{B} \cdot d\vec{l}$ has two different values which is not allowed.

In ~~a~~ analogy to Faraday's Law, one might consider fixing this inconsistency with the time rate of change of electric flux.

The electric flux through surface S_1 is

$$\Phi_e = EA$$

where A is the area of the plate and E the electric field.

$$\frac{d\Phi_e}{dt} = A\frac{dE}{dt} \qquad E = \frac{\sigma}{\varepsilon_0}$$

$$= \frac{d}{dt} A\sigma/\varepsilon_0$$

$$= \frac{1}{\varepsilon_0}\frac{dQ}{dt} = \frac{I}{\varepsilon_0}$$

So $\varepsilon_0 \dfrac{d\Phi_e}{dt}$ is zero on surface 2 and

I on surface 1, just what we need.

Dfn Displacement Current (I_d) A quantity with dimensions of current needed to complete Ampere's Law.

$$I_d = \varepsilon_0 \frac{d\Phi_e}{dt}$$

Ampere's Law (Complete)

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \left(I_{enc} + I_d \right)$$

$$= \mu_0 \int_S \vec{J} \cdot d\vec{a} + \mu_0 \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{a}$$

Differential Form

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

An alternate method of identifying the need to fix Ampere's Law.

Ampere's Law

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Take the divergence; the divergence of a curl is always zero

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J} = 0$$

For magnetostatics, $\nabla \cdot \vec{J} = 0$ and everything is good, but in general

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \text{(continuity eqn)}$$

$$\Rightarrow \nabla \cdot \vec{J} \neq 0$$

but $\nabla \cdot \vec{J} + \dfrac{\partial \rho}{\partial t} = 0.$

So maybe

$$\nabla \cdot \nabla \times \vec{B} = \mu_0 \left(\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right)$$

Use Gauss' Law

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \Rightarrow \rho = \varepsilon_0 \nabla \cdot \vec{E}$$

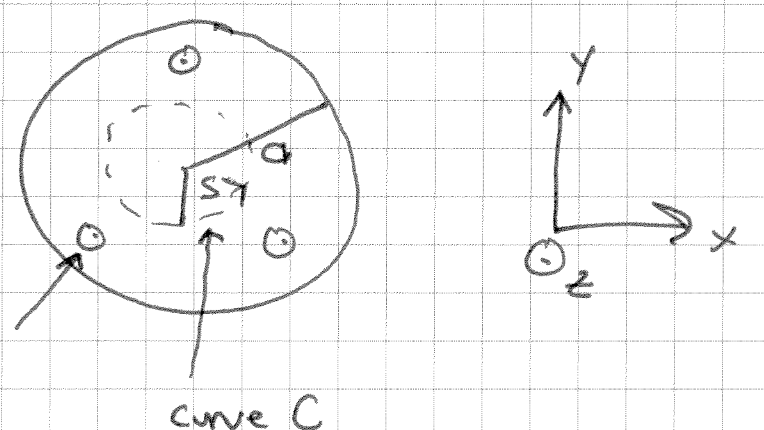
$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \left(\nabla \cdot \vec{J} + \varepsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t} \right)$$

so Ampere's Law is fine if

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$\Rightarrow$ The displacement current acts as a source in Ampere's Law just like a real current.

Ex A cylindrical $s < a$ contains a time varying electric field $\vec{E} = E_0 e^{-t/\tau} \hat{z}$



The electric flux through a circular surface of radius $s < a$ is

$$\Phi_e = EA = E \pi s^2$$
$$= E_0 \pi s^2 e^{-t/\tau}$$

$\Rightarrow$ Note, convention for positive normal used

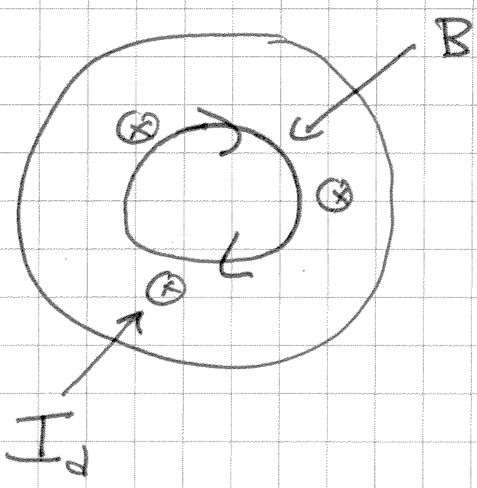
Displacement Current

$$I_d = \varepsilon_0 \frac{d\Phi_e}{dt}$$

$$= -\frac{E_0 \pi s^2}{\tau} e^{-t/\tau}$$

$\Rightarrow$ - sign indicates I_d directed into the page.

Since I_d is cylindrically symmetry, the magnetic field generated by I_d is circular about the axis. The Right Hand rule still gives the direction



Ampere $\oint \vec{B} \cdot d\vec{l} = \mu_0 (I_{enc} + I_d) = \mu_0 I_d$

$$= 2\pi s B$$

$$B = \frac{\mu_0 I_d}{2\pi s} = \frac{\mu_0}{2\pi s}\left(-\frac{\varepsilon_0 E_0 \pi s^2}{\tau} e^{-t/\tau}\right)$$

$$\vec{B} = \frac{-\varepsilon_0 \mu_0 E_0 s}{2\tau} e^{-t/\tau}\,\hat{\phi} \qquad s<a$$

Outside Region $s>a$

$$\Phi_e = \pi a^2 E$$

$$I_d = \varepsilon_0 \frac{d\Phi_e}{d\tau} = \frac{-\pi a^2 E_0}{\tau} e^{-t/\tau}$$

$$\vec{B} = \frac{\mu_0 I_d}{2\pi s}\hat{\phi} = \frac{-a^2 E_0}{2 s\tau} e^{-t/\tau}\,\hat{\phi}$$

We can also define a displacement current density

$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

For the above system

$$\vec{J}_d = -\frac{\varepsilon_0}{\tau} E_0 e^{-t/\tau}\,\hat{z} \qquad s<a$$

and then

$$I_d = \int_S \vec{J}_d \cdot d\vec{a} = J_d A$$

$$= \frac{-\varepsilon_0 \pi s^2 E_0}{T} e^{-t/T}$$