Magnetostatics

Magnetostatics is the study of systems with constant current, so

$$\frac{\partial \rho}{\partial t} = 0 \qquad \nabla \cdot \vec{J} = 0$$

Biot-Savart Law - The magnetic field of a point $\vec{r}$ due to a distribution of current $\vec{J}(\vec{r}\,')$ is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}\,') \times \hat{r}''}{(r'')^2} \, d\tau'$$

- $\vec{r}\,'' = \vec{r} - \vec{r}\,'$ as before
- Permeability of Free Space

$$\mu_0 = 4\pi \times 10^{-7} \ \frac{\text{Tm}}{\text{A}}$$

- Units of $\vec{B}$

$$1 \text{ Tesla} = 1\text{T} = 1 \frac{\text{Ns}}{\text{Am}} = \frac{\text{N}}{\text{Am}}$$

Other Forms

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S \frac{\vec{K} \times \hat{r}''}{(r'')^2}\, da'$$ sheet current

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_C \frac{\vec{I} \times \hat{r}''}{(r'')^2}\, dl'$$ linear or curve current

$$= \frac{\mu_0}{4\pi} \int I \frac{d\vec{l}\,' \times \hat{r}''}{(r'')^2}$$

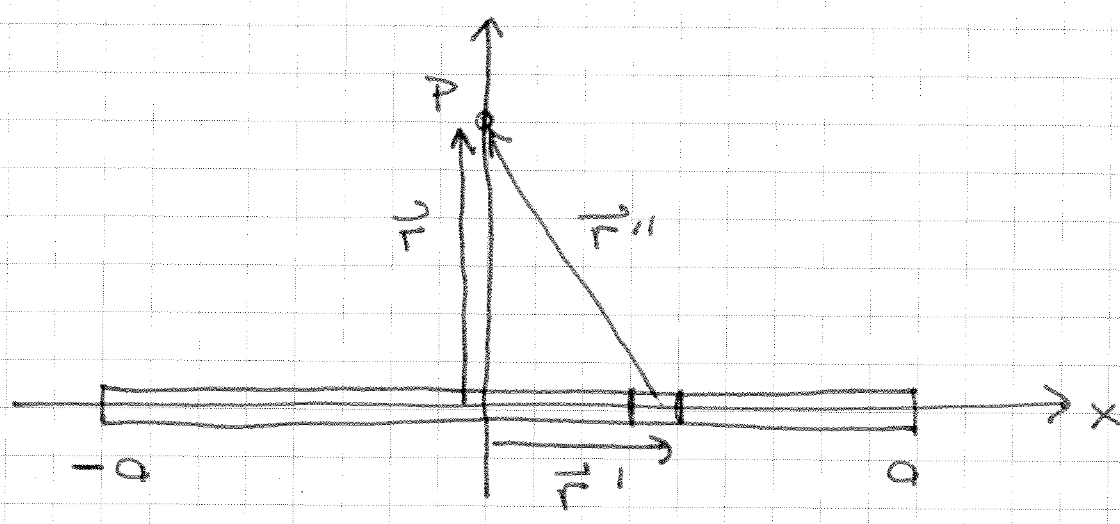
and very approximately, if $v \ll c$

$$\vec{B} = \frac{\mu_0}{4\pi} \quad q\, \frac{\vec{v} \times \hat{r}''}{(r'')^2}$$

for q moving at $\vec{v}$

$\Longrightarrow$ Note, not magnetostatic

Ex Compute field of straight wire segment from $-a$ to a along the x-axis carrying current I. Compute field at $\vec{r}_p = y\hat{y}$.



Source Point $\quad \vec{r}' = (x', 0, 0)$

Field Point $\quad \vec{r} = (0, y, 0)$

Displacement $\quad \vec{r}'' = \vec{r} - \vec{r}' = (-x', y, 0)$

$$r'' = \sqrt{(x')^2 + y^2}$$

Biot-Savart Law

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_c \frac{d\vec{l}' \times \hat{r}''}{(r'')^2}$$

$$= \frac{\mu_0 I}{4\pi} \int_c \frac{d\vec{l}' \times \vec{r}''}{(r'')^3}$$

$$d\vec{l}' = dx'\,\hat{x}$$

$$d\vec{l}' \times \vec{r}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & 0 & 0 \\ -x' & y & 0 \end{vmatrix}$$

$$= y\,dx'\,\hat{z}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_{-a}^{a} \frac{y\,\hat{z}\,dx'}{(x'^2+y^2)^{3/2}}$$

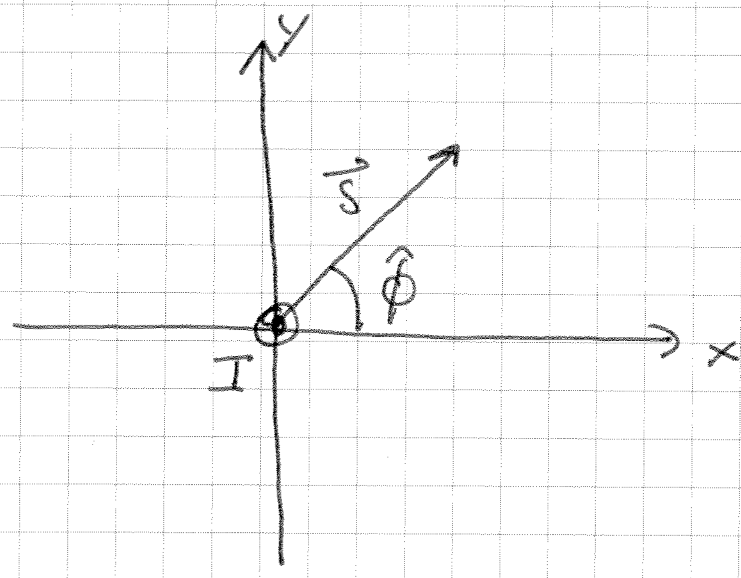
$$= \frac{\mu_0 I y\,\hat{z}}{4\pi} \int_{-a}^{a} \frac{dx'}{(x'^2+y^2)^{3/2}}$$

$$= \frac{\mu_0 I}{2\pi y} \left(\frac{a}{\sqrt{y^2+a^2}} \right) \hat{z}$$

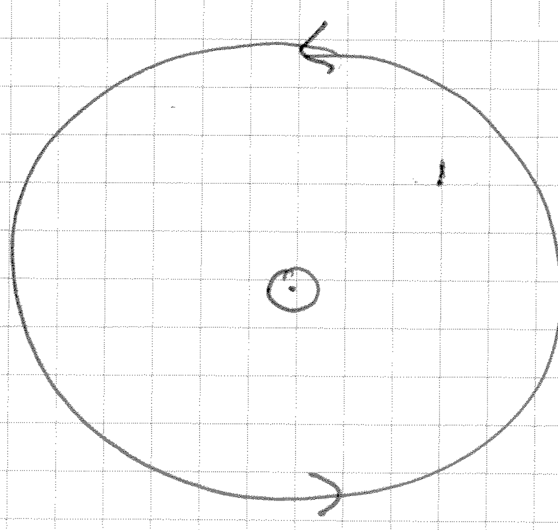
Ex Field of Infinite Wire

Let $a \to \infty$, $\vec{B} \to \dfrac{\mu_0 I}{2\pi y}\hat{z}$

More generally, if we let $\vec{I} = I\hat{z}$

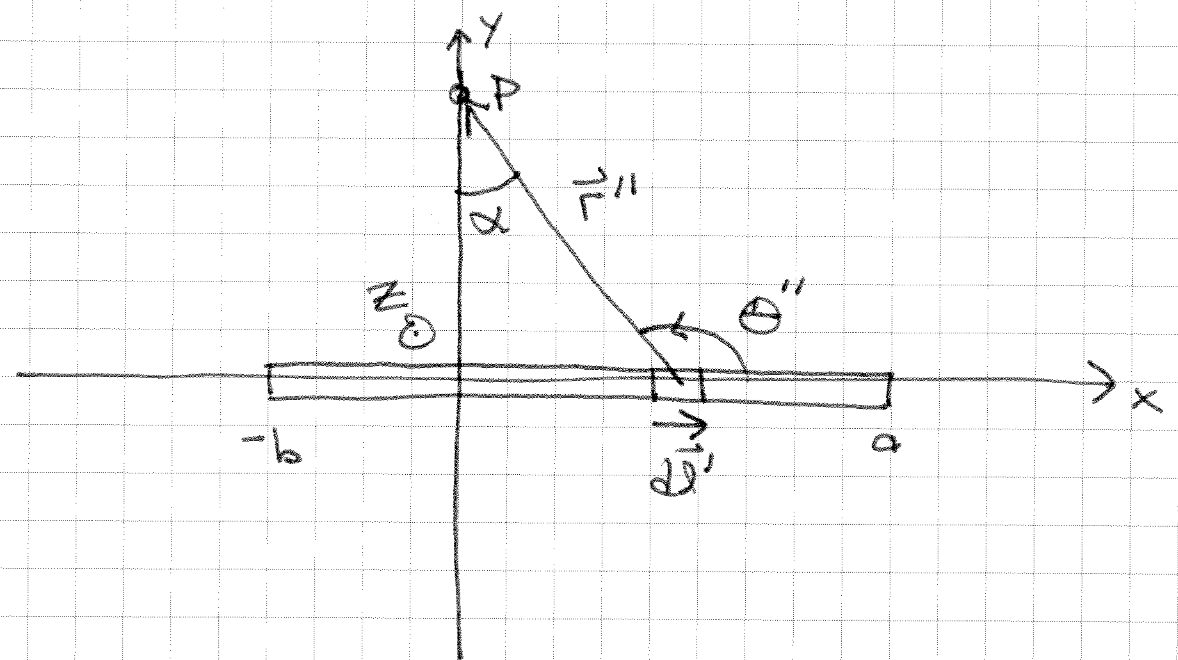


$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$



<u>RHR for Wire</u> - Grab the wire with your right hand with your thumb pointing in the direction of current. Your fingers curl in the direction of the field.

<u>Ex</u> Alternate solution for short wire



$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}''}{(r'')^2}$$

By Right Hand Rule, $\vec{B}$ points out of page.

$$\cancel{d\vec{B}} = |d\vec{B}|\,\hat{z} = d\vec{B}$$

$$|d\vec{B}| = \frac{\mu_0 I}{4\pi r''^2} |d\vec{l}'||\hat{r}''| = \frac{\mu_0 I}{4\pi r''^2} dl' \sin\theta''$$

$$\theta'' = \alpha + \pi/2$$

$$d\ell' = dx'$$

$$\cos\alpha = \frac{y}{r''} = \sin\theta''$$

$$r'' = \frac{y}{\sin\theta''}$$

$$|d\vec{B}| = \frac{\mu_0 I \, dx' \sin\theta''}{4\pi (y/\sin\theta'')^2} = \frac{\mu_0 I}{4\pi y^2} \sin^3\theta'' dx'$$

$$\sin\alpha = \frac{x'}{r''} = \sin(\theta'' - \pi/2)$$

$$= -\cos(\theta'')$$

$$x' = -r''\cos\theta'' = -y \frac{\cos\theta''}{\sin\theta''}$$

$$= -y \cot\theta''$$

$$dx' = -y \, d(\cot\theta'') =$$

$$= y \csc^2\theta'' d\theta'' = \frac{y \, d\theta''}{\sin^2\theta''}$$

$$|d\vec{B}| = \left| \frac{\mu_0 I}{4\pi y} \sin\theta'' d\theta'' \right|$$

$$|\vec{B}| = \frac{\mu_0 I}{4\pi y} \left| \int_{\theta''_b}^{\theta''_a} \sin\theta'' d\theta'' \right|$$

$$= \frac{\mu_0 I}{4\pi y} \left| -(\cos\theta''_a - \cos\theta''_b) \right|$$

$$= \frac{\mu_0 I}{4\pi y} \left| \cos\left(\alpha_b - \frac{\pi}{2}\right) - \cos\left(\alpha_a - \frac{\pi}{2}\right) \right|$$

$$= \frac{\mu_0 I}{4\pi y} \left| \sin\alpha_b - \sin\alpha_a \right|$$

$$\vec{B} = \frac{\mu_0 I}{4\pi y} (\sin\alpha_a - \sin\alpha_b)\, \hat{z}$$

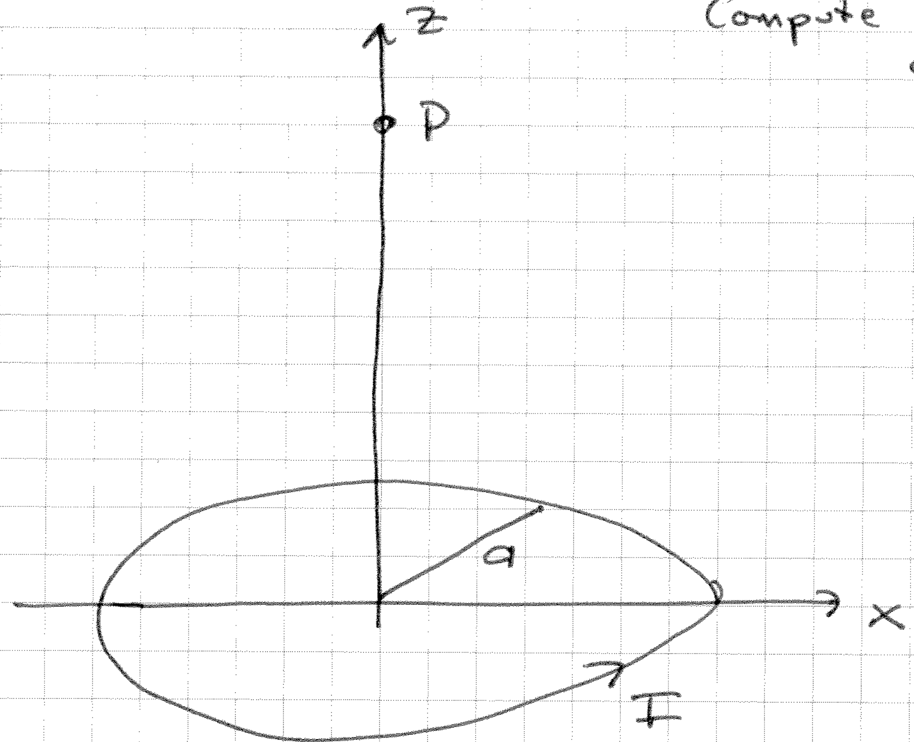
picking the orientation α_a as positive and α_b as negative.

Let ~~*~~ $\alpha_a \to \pi/2$, $\alpha_b \to -\pi/2$

$$\vec{B} \to \frac{\mu_0 I}{2\pi y} \hat{z}$$

Ex Ring of Current in x-y plane

Compute field on axis.



$$\vec{I} = I\hat{\phi}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int_C \frac{\vec{I}(\vec{r}\,') \times \hat{r}'' \, dl'}{(r'')^2}$$

$$= \frac{\mu_0 I}{4\pi}\int_C \frac{d\vec{l}\,' \times \hat{r}''}{(r'')^2}$$

$\vec{I} = I\hat{n}$
$d\vec{l}\,' = \hat{n}' dl$

$$d\vec{l}\,' = a\,d\phi'\,\hat{\phi}'$$

Field Point $\vec{r} = (0, \cancel{\phi}, z) = z\hat{z}$

Source Point $\vec{r}\,' = s'\hat{s}'$

Displacement Vector $\vec{r}'' = \vec{r} - \vec{r}\,'$

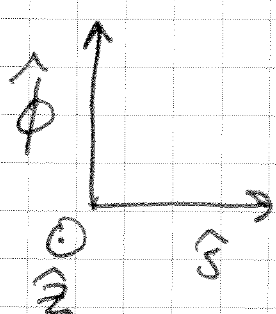
$$\vec{r}'' = z\hat{z} - s'\hat{s}' = z\hat{z} - a\hat{s}'$$

Length $r'' = \sqrt{a^2 + s'^2}$

$$d\vec{\ell}\,' \times \vec{r}'' = a\,d\phi'\,\hat{\phi}' \times \left(z\hat{z} - a\hat{s}' \right)$$

$$= a z\, d\phi' \left(\hat{\phi}' \times \hat{z} \right) - a^2 d\phi' \left(\hat{\phi}' \times \hat{s}' \right)$$

Cylindrical Coordinates Right Handed Coordinate System



$$\hat{\phi} \times \hat{z} = \hat{s} \qquad \hat{\phi} \times \hat{s} = -\hat{z}$$

$$d\vec{l}' \times \vec{r}'' = a^2 \hat{z}\, d\phi' + a z\, d\phi' \hat{s}'$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{d\phi' \hat{s}' a z}{(z^2+a^2)^{3/2}} \equiv I_1$$

$$+ \frac{\mu_0 I}{4\pi} \int \frac{d\phi' a^2 \hat{z}}{(z^2+a^2)^{3/2}} \equiv I_2$$

<u>First Integral</u>

$$\int_0^{2\pi} \frac{a z\, d\phi' \hat{s}'}{(z^2+a^2)^{3/2}} = \frac{a z}{(z^2+a^2)^{3/2}} \int_0^{2\pi} d\phi\, \hat{s}'$$

$$= \frac{a z}{(z^2+a^2)^{3/2}} \int_0^{2\pi} (\cos\phi' \hat{x} + \sin\phi' \hat{y})\, d\phi'$$

$$= 0$$

<u>Second Integral</u>

$$\int_0^{2\pi} \frac{d\phi' a^2 \hat{z}}{(z^2+a^2)^{3/2}} = \frac{a^2 \hat{z}}{(z^2+a^2)^{3/2}} \int_0^{2\pi} d\phi'$$

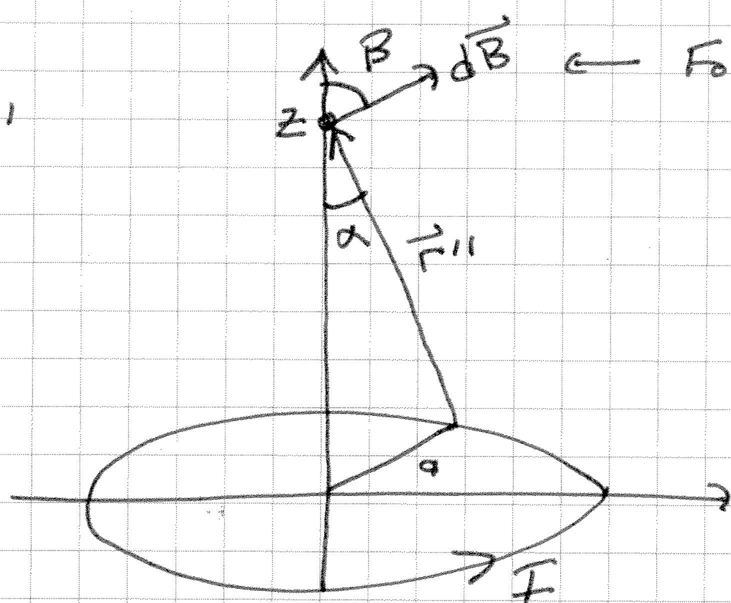
$$= \frac{2\pi a^2 \hat{z}}{(z^2+a^2)^{3/2}}$$

So

$$\vec{B} = \frac{\mu_0 I}{4\pi} \cdot \frac{2\pi a^2 \hat{z}}{(z^2+a^2)^{3/2}}$$

$$= \frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}}$$

Alternately,



← Found by RHR

$$\alpha + \beta + \pi/2 = \pi \qquad \beta = \frac{\pi}{2} - \alpha$$

$$\sin\alpha = \frac{a}{r''}$$

By symmetry, only z component of $\vec{B}$ survives

$$|d\vec{B}_z| = \frac{\mu_0 I}{4\pi} \frac{d\ell}{r''^2} \cos\beta$$

$$= \frac{\mu_0 I}{4\pi} \frac{d\ell}{r''^2} \sin\alpha$$

Note, no $\sin\theta$ from cross-product becouse $\vec{r}''$ and $d\vec{\ell}'$ are $\perp$. $\Rightarrow \sin\frac{\pi}{2} = 1$

$$\vec{B} = \hat{z}\int |dB_z| = \frac{\mu_0 I}{4\pi} \frac{\sin\alpha}{(r'')^2} \int_0^{2\pi} d\ell$$

$d\ell = a\,d\phi$

$$= \frac{\mu_0 I}{4\pi} \frac{\sin\alpha}{(r'')^2} 2\pi a \;\hat{z}$$

$$= \frac{\mu_0 I a}{2} \frac{\sin\alpha}{r''} \hat{z}$$

$$= \frac{\mu_0 I a^2}{2 r''^3} \hat{z} = \frac{\mu_0 I}{2} \frac{a^2}{(z^2+a^2)^{3/2}} \hat{z}$$