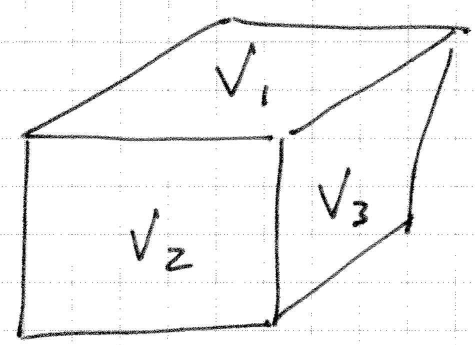
# Boundary Value Problems

Physically, there are many ways to set up an electrostatic system in a region of space.

I. We can enclose the region in a set of conductors and fix the potential on each conductor.




This can be accomplished by connecting a different power supply to each face.

II. We can fix some non-zero charge density $\rho$ in place either in the cube about or in an infinitely large sphere where we set the potential on the sphere to zero.

III. We can place some conductors each at potential $V_i$ and some fixed charge $\rho$ in the region and then specify $V_i$ on the cube or at $\infty$.

IV. We can place some charge $Q_i$ on a set of conductors, place some fixed charge $\rho$, specify $V_i$ on the cube or at $\infty$.

$\Rightarrow$ Note, specifying $Q_i$ is the same as specifying $\int \nabla V \cdot d\vec{a}$ on the conductor.

V. We can specify the field at the surface of the region and the charge on a set of conductors in the region.

$\Rightarrow$ Note, specifying the field is the same as specifying $\nabla V$.

$\Rightarrow$ For any system we can actually build, the universe will produce a unique $V$, $\vec{E}$, $\rho$.

<u>Uniqueness Thms</u> When is $\nabla^2 V = -\rho/\varepsilon_0$ uniquely determined?

<u>Dirichlet Problem</u> The electric potential is uniquely determined if:

1) The location and size of all charge in the volume $V$ is fixed.

2) The potential is fixed on the surface bounding $V$.

<u>Neumann Problem</u> The electric potential is uniquely determined if:

1) The location of all charge is fixed in a volume $V$.

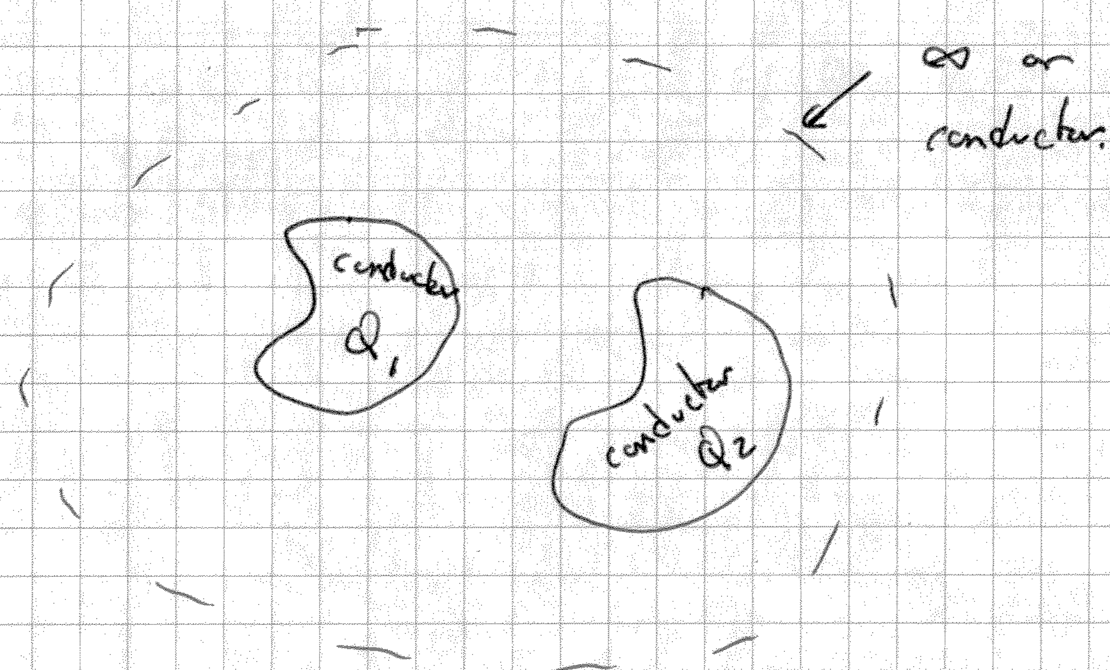
2) The normal derivative of $V$, $\frac{\partial V}{\partial n}$, is specified on the surface bounding $V$.

$\Rightarrow$ If bounding surface is conducting, (2) requires specifying surface charge.

$\Rightarrow$ These conditions can be mixed with part of the bounding surface specifying $V$ and part $\frac{\partial V}{\partial n}$ and a unique solution will still exist.

## Griffith's Uniqueness Thm –

If a volume $V$ is enclosed by conductors or has a bounding surface at $\infty$ and the charge density $\rho$ is fixed in the volume, then the potential is uniquely determined if the total charge on any conductor in the volume is given.



So what do we want to do with these systems?

Find $\vec{E}$ s.t.

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \qquad \nabla \times \vec{E} = 0$$

If $\vec{E} = -\nabla V$, the curl eqn is automatically satisfied, so the full problem is given by Poisson's Eqn

$$\nabla^2 V = -\rho/\varepsilon_0$$

---

<u>Dfn Homogeneous</u> – An ~~equalition~~ equation is homogeneous if two solutions $f_1$ and $f_2$ imply that $f_1 + f_2$ is also a solution.

Poisson's Eqn is NOT homogeneous, but Laplace's Eqn $\nabla^2 V = 0$, Poisson's eqn in charge free regions is.

<u>Strategy</u> - (1) Find any solution to Poisson's eqn in the region ignoring the boundary conditions. Call this solution the particular solution $V_p$.

(2) Find all solutions to Laplace's eqn in the region, $f_i$.

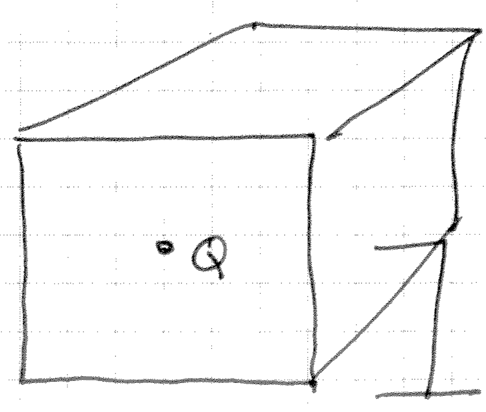
(3) Use a combination of $f_i$ to satisfy the boundary conditions. Call this combination the homogeneous solution

$$V_h = \sum a_i f_i$$

$$V = V_p + V_h$$

---

<u>Example</u> - Find the potential of a point charge in a grounded cube




$\Rightarrow V = 0.$

The particular solution, that does not satisfy $V=0$ on the cube is

$$V_p = \frac{KQ}{r}$$

$$\nabla^2 V = \frac{-Q\delta^3(\vec{r})}{\varepsilon_0}$$

Find functions $V_h = \sum_i a_i f_i(x,y,z)$

where $\nabla^2 f_i = 0$ and $a_i$ are chosen to make $V=0$ on cube.

---

<u>Property of Laplace's Eqn</u>

The potential at a point $\vec{r}$ is the average of the potential over a sphere of any radius $R$

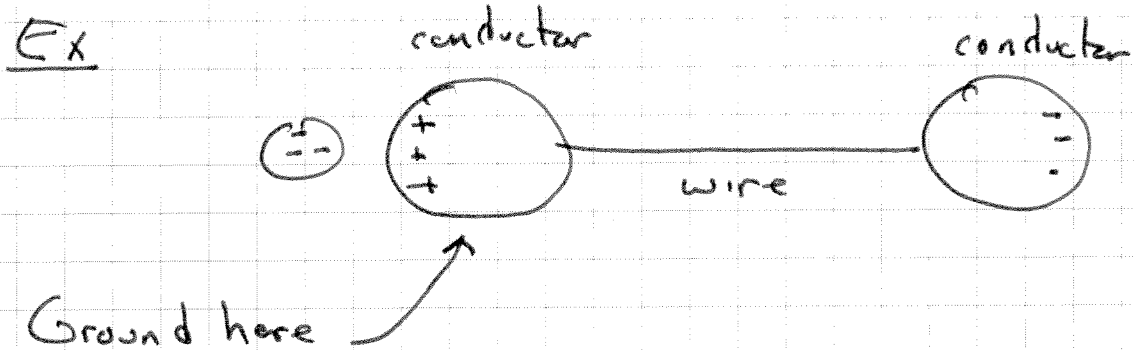
$$V(\vec{r}) = \frac{1}{4\pi R^2}\int_{\text{sphere}} V(\vec{r}\,')\,da'$$

$\Longrightarrow$ $V$ can have no local minima or maxima; all extrema must be on the boundaries.

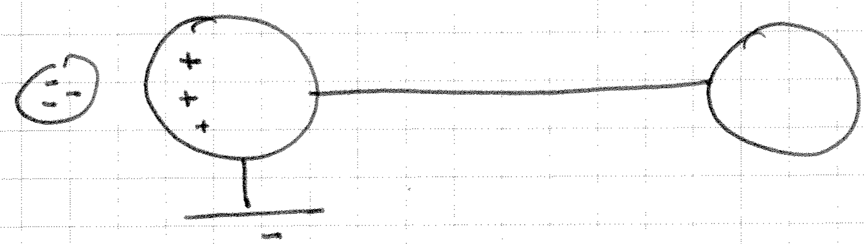
$\Rightarrow$ If there is a lower energy state for a system of charge, the system will find it.

$\Rightarrow$ There are no metastable electrostatic systems

<u>Ex</u>




$$\Downarrow$$



$\Rightarrow$ $-$ go to ground to find minimum of energy.