Work, Force, Energy, and Capacitance with Dielectrics

Gauss' Law for Displacement

$$\nabla \cdot \vec{D} = \rho_f \qquad \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

If the dielectric is linear

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_r \varepsilon_0 \vec{E} = \kappa \varepsilon_0 \vec{E}$$

$$\varepsilon_r = \kappa = 1 + \chi_e$$

Electrostatic Boundary Conditions

$$\nabla \cdot \vec{D} = \rho_f \implies \vec{D}_2 \cdot \hat{n} - \vec{D}_1 \cdot \hat{n} = \sigma_f$$

$$\nabla \times \vec{E} = 0 \implies \vec{E}_2 \cdot \hat{t} = \vec{E}_1 \cdot \hat{t}$$

If the dielectric completely fills the fieldspace and there are no interfaces

$$\nabla \cdot \vec{D} = \rho_f \quad , \quad \nabla \times \vec{D} = 0$$

which are the same equations we would have to solve if the dielectric did not exist. Solve them without the dielectric to yield

$$\vec{D} = \varepsilon_0 \vec{E}_0$$

where $\vec{E}_0$ is the field the free charge would produce without the dielectric.

This must also be $\vec{D}$ with the dielectric

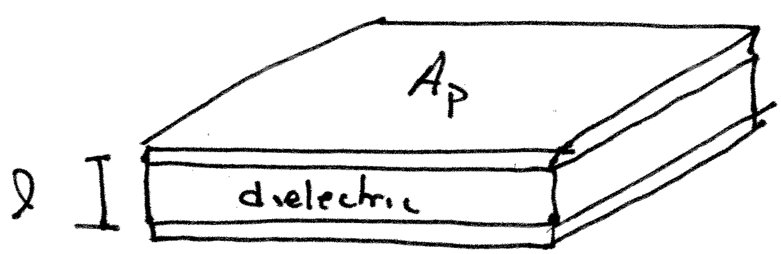
$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E}_0$$

where $\vec{E}$ is the field with the dielectric.

$$\vec{E} = \frac{\vec{E}_0}{\varepsilon_r} = \frac{\vec{E}_0}{\kappa}$$

$\Rightarrow$ The effect of the dielectric is to reduce the field by κ.

Ex Parallel-plate capacitor with plate area A_p and separation l completely filled with dielectric with dielectric constant κ.



$\Rightarrow$ Assume no fringing $\Rightarrow$ can treat field as field of infinite planes.

Add $+Q$ to one plate, $-Q$ to other producing charge densities

$$\sigma = \pm \frac{Q}{A_p}$$

Without the dielectric, this produces a field

$$|\vec{E}_0| = \frac{\sigma}{\varepsilon_0}$$

and a potential difference between the plates

$$\Delta V = -\int_0^{\ell} \vec{E} \cdot d\vec{\ell} = \pm |E_0| \ell = \pm \frac{\sigma \ell}{\varepsilon_0}$$

The capacitance is then

$$C_0 = \frac{Q}{|\Delta V|} = \frac{\sigma A_p}{|\Delta V|} = \frac{\varepsilon_0 A_p}{\ell}$$

With the dielectric, the field is reduced by κ

$$\vec{E}_\kappa = \text{Field with dielectric} = \frac{\vec{E}_0}{\kappa}$$

The potential difference is reduced by κ

$$\Delta V_\kappa = -\int_0^{\ell} \vec{E}_\kappa \cdot d\vec{\ell} = \pm |\vec{E}_\kappa| \ell$$

$$= \pm \frac{\sigma \ell}{\kappa \varepsilon_0}$$

and the capacitance is increased by κ

$$C_\kappa = \frac{Q}{|\Delta V_\kappa|} = \frac{\sigma A_p}{|\Delta V_\kappa|} = \kappa \left(\frac{\varepsilon_0 A_p}{\ell} \right) = \kappa C_0$$

The work required to charge the capacitor is still

$$W = \int dW = \int \Delta V \, dQ = \int_0^Q \frac{Q\,dQ}{C}$$

$$= \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C(\Delta V)^2$$

$= U$ the energy stored in the capacitor

The energy stored in the parallel-plate capacitor without the dielectric is

$$U = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}C\ell^2\left(\frac{\Delta V}{\ell}\right)^2$$

$$= \frac{1}{2}C\ell^2 E_0^2$$

$$= \frac{1}{2}\left(\frac{\varepsilon_0 A_P}{\ell}\right)\ell^2 E_0^2$$

$$= \left(\frac{1}{2}\varepsilon_0 E_0^2\right)(A_P\ell)$$

where $A_P\ell$ is the volume $=V$ of the region between the plates

The energy density of the field is then ~~E~~

$$u = \frac{U}{Vol} = \frac{1}{2}\varepsilon_0 E_0^2$$

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With the dielectric, the energy stored in the capacitor is

$$U = \frac{1}{2} C_K (\Delta V)^2 = \frac{1}{2} d^2 C_K E_K^2$$

$$= \frac{1}{2}\left(\frac{K \varepsilon_0 A_P}{d}\right) d^2 E^2$$

E_K is field with dielectric

The energy density is

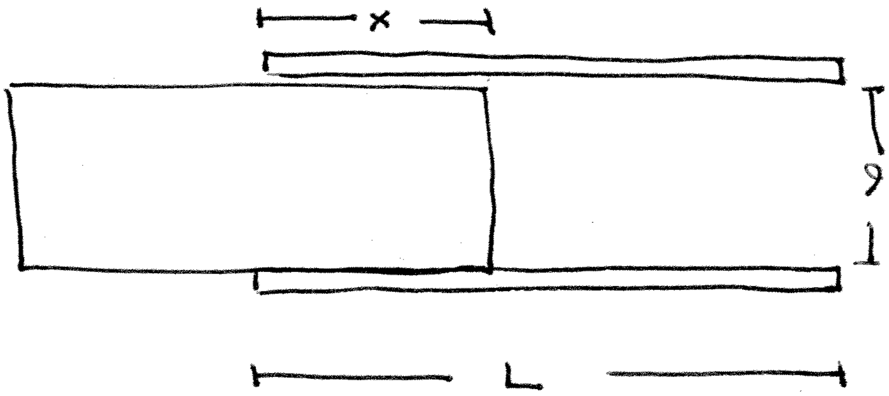
$$u = \frac{U}{\text{Volume}} = \frac{1}{2} K \varepsilon_0 E^2$$

$$D = K \varepsilon_0 E$$

$$\boxed{u = \frac{1}{2} \vec{D} \cdot \vec{E}}$$

Energy density of field with dielectric

Force on Dielectric as it is Inserted in Capacitor

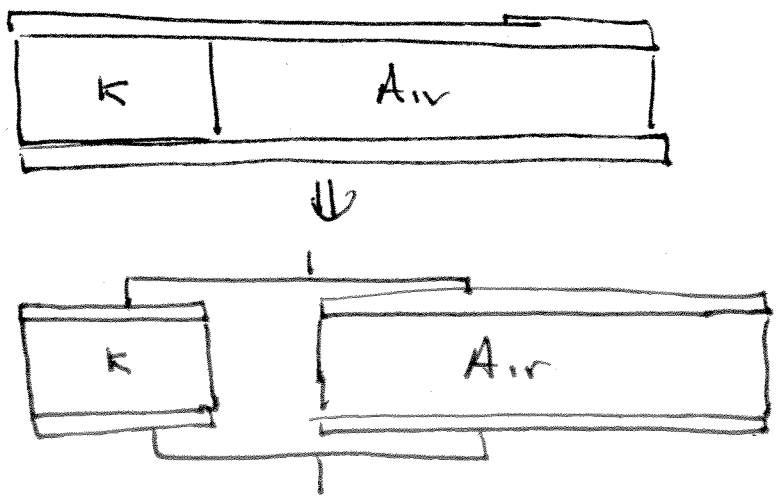


Case I - ~~Connected~~ Disconnected from battery, but charged $\Rightarrow$ Q fixed

Force $F = -\dfrac{dU}{dx} = -\dfrac{d}{dx}\dfrac{Q^2}{2C}$

$$= -\frac{Q^2}{2}\frac{d}{dx}\frac{1}{C}$$

$$= \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}$$



Parallel capacitors

Note, ΔV is the same for both capacitors so E is the same and the electrostatic boundary conditions are satisfied.

Parallel Capacitors

$$C_{eq} = C_1 + C_2$$

$$= \kappa \frac{\varepsilon_0 A_1}{\ell} + \frac{\varepsilon_0 A_2}{\ell}$$

Let the plates be square, $A_p = L^2$

$$C_{eq} = \frac{\varepsilon_0 L x \kappa}{\ell} + \frac{\varepsilon_0 L (L-x)}{\ell}$$

$$\frac{dC_{eq}}{dx} = \frac{\kappa \varepsilon_0 L}{\ell} - \frac{\varepsilon_0 L}{\ell} = \frac{\varepsilon_0 L}{\ell}(\kappa - 1)$$

$$= \frac{C_0}{L}(\kappa - 1) \qquad C_0 = \frac{\varepsilon_0 L^2}{\ell}$$

air filled capacitance

$$C_{eq} = \frac{L\varepsilon_0}{l}\left(kx + L - x\right)$$

$$= C_0\left((k-1)\frac{x}{L} + 1\right)$$

$$F = \frac{1}{2}\frac{Q^2}{C_{eq}^2}\frac{dC_{eq}}{dx}$$

$$= \frac{1}{2}Q^2 \frac{\frac{C_0}{L}(k-1)}{C_0^2\left((k-1)\frac{x}{L}+1\right)^2}$$

$$= \frac{1}{2}\frac{Q^2}{C_0 L} \cdot \frac{k-1}{\left((k-1)\frac{x}{L}+1\right)^2}$$

$$= \frac{1}{2}\frac{Q^2}{C_0 L} \cdot \frac{\chi_e}{\left(\chi_e\frac{x}{L}+1\right)^2}$$

$F > 0$, dielectric pulled into capacitor.

<u>Case II</u> - Connected to battery - $\Delta V_{constant}$

$\Rightarrow$ Must include work done by battery

$$dU = dU_{cap} - dQ\, V_{batt}$$

Total Energy is the energy stored in the capacitor and the energy stored in battery. Transferring charge to the capacitor requires work and lowers battery energy.

$$\frac{dU_{cap}}{dx} = \underbrace{\frac{1}{2}(\Delta V)^2}_{\text{constant}} \frac{dC}{dx}$$

$$\frac{dU_{batt}}{dx} = -\Delta V \frac{dQ}{dx} = -\Delta V\left(\Delta V \frac{dC}{dx}\right)$$

$$= -(\Delta V)^2 \frac{dC}{dx}$$

$$F = -\frac{dU_{sys}}{dx} = -\Big[\underbrace{\frac{1}{2}(\Delta V)^2 \frac{dC}{dx}}_{\text{capacitor}} \underbrace{-(\Delta V)^2 \frac{dC}{dx}}_{\text{battery}}\Big]$$

$$= +\frac{1}{2}(\Delta V)^2 \frac{dC}{dx}$$

We've already worked out dC/dx

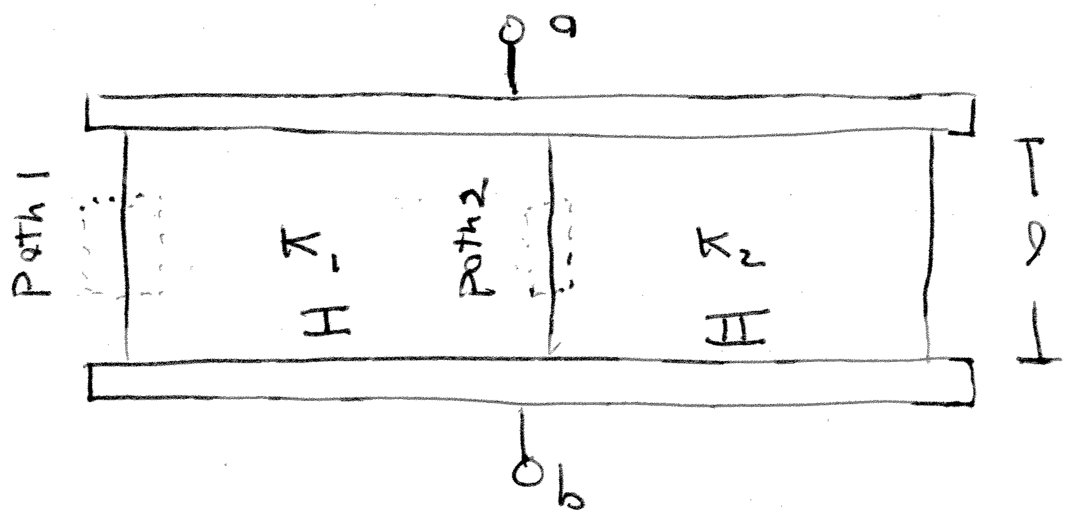
$$F = \frac{1}{2}(\Delta V)^2 \frac{dC}{dx}$$

$$= \frac{1}{2}(\Delta V)^2 \frac{C_0}{L}(\kappa - 1)$$

$$= \frac{1}{2}(\Delta V)^2 \frac{\chi_e C_0}{L}$$

$F > 0$, dielectric still pulled into capacitor.

Ex Capacitor with two dielectrics



Electric Field

$$E_I = \frac{\Delta V}{l} \qquad E_2 = \frac{\Delta V}{l}$$

Electrostatic B.C. satisfied for path 2 but not path 1. If system extends to ∞, fields are exact; if not we must assume the contribution of the fringe region at the outside edge is small.

The fields are proportional to the charge densities.

$$E_1 = \frac{\sigma_1}{\kappa_1 \varepsilon_0} \qquad\qquad E_2 = \frac{\sigma_2}{\varepsilon_0 \kappa_2}$$

$$\sigma_1 = \kappa_1 \varepsilon_0 E_1 \qquad\qquad \sigma_2 = \kappa_2 \varepsilon_0 E_2$$

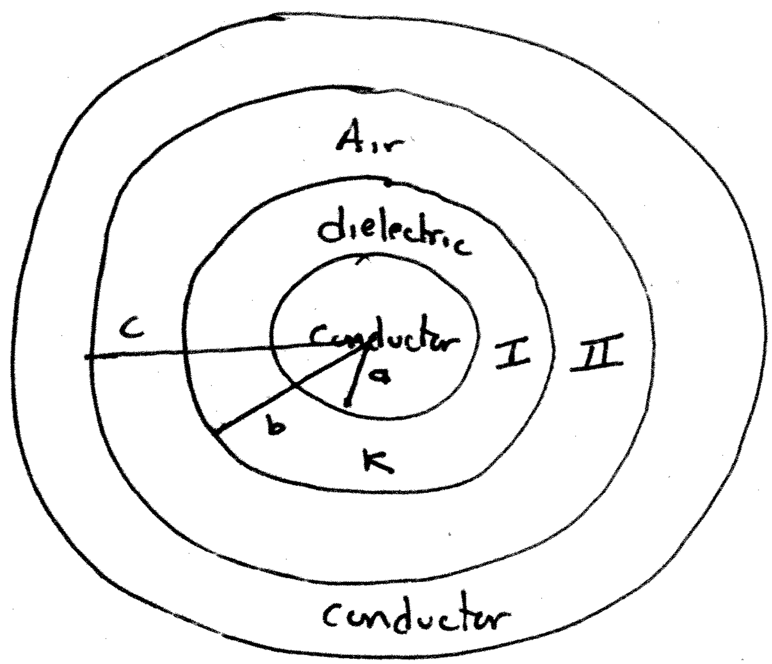
$$= \kappa_1 \varepsilon_0 \frac{\Delta V}{l} \qquad\qquad = \kappa_2 \varepsilon_0 \frac{\Delta V}{l}$$

Note $\sigma_1 \neq \sigma_2$

Note σ_1, σ_2 free charge densities

Other Geometries (spherical)

Capacitance between inner and outer conductor



Note, dielectric does not completely space between conductors so $C_\kappa \neq \kappa C_0$

Introduce Q on inner conductor, $-Q$ on outer conductor.

Displacement Field

$$\Phi_d = 4\pi r^2 D = Q_{f,enc} = Q$$

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$$

Check Electrostatic B.C. at dielectric/air interface

(1) $\sigma_f = 0$, $\vec{D}_2 \cdot \hat{n} - \vec{D}_1 \cdot \hat{n} = 0$ ✓

2) Field radial, $\vec{E}_2 \cdot \hat{t} - \vec{E}_1 \cdot \hat{t} = 0$ ✓

Solution exact

In Air (Region II)

$$\vec{D} = \varepsilon_0 \vec{E}_{II}$$

$$\vec{E}_{II} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

In Dielectric (Region I)

$$\vec{D} = \kappa \varepsilon_0 \vec{E}_I$$

$$\vec{E}_I = \frac{\vec{D}}{\kappa\varepsilon_0} = \frac{Q}{4\pi\varepsilon_0 \kappa r^2}\hat{r}$$

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Potential Difference

$$\Delta V_{I} = -\int_a^b \vec{E}_I \cdot d\vec{l}$$

$$= -\frac{Q}{4\pi\varepsilon_0 \kappa}\left(\frac{1}{a} - \frac{1}{b}\right) < 0$$

$$\Delta V_{II} = -\int_b^c \vec{E}_{II} \cdot d\vec{l}$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{c}\right)$$

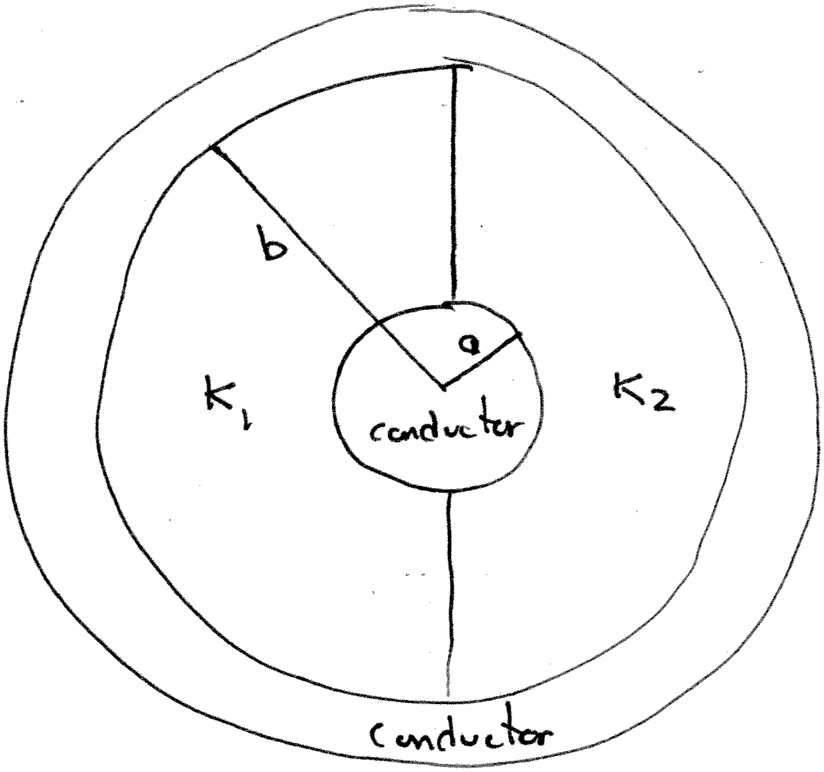
Total Potential Difference

$$\Delta V = \Delta V_I + \Delta V_{II} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{c} + \frac{1}{a\kappa} - \frac{1}{b\kappa}\right)$$

Capacitance

$$C = \frac{Q}{|\Delta V|} = \frac{4\pi\varepsilon_0}{\frac{1}{b} - \frac{1}{c} + \frac{1}{a\kappa} - \frac{1}{b\kappa}}$$

Ex Spherical system with two dielectrics



Place $+Q$ on inner conductor, $-Q$ on outer.

⇒ Note, system is NOT spherically symmetric.

⇒ If either dielectric completely filled the field space we would find

$$\vec{D} = K_1 \varepsilon_0 \vec{E}_1 \qquad \text{or} \qquad \vec{D} = K_2 \varepsilon_0 \vec{E}_2$$

$$\vec{E}_1 = \frac{Q_1}{4\pi\varepsilon_0 K_1 r^2}\hat{r} \qquad \vec{E}_2 = \frac{Q_2}{4\pi\varepsilon_0 K_2 r^2}\hat{r}$$

Since these are solutions to the appropriate differential equations we can try to fit them together and rely on uniqueness.

$$\Delta V_1 = -\int \vec{E}_1 \cdot d\vec{l} = \frac{-Q_1}{4\pi\varepsilon_0 K_1}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\Delta V_2 = -\int \vec{E}_2 \cdot d\vec{l} = \frac{-Q_2}{4\pi\varepsilon_0 K_2}\left(\frac{1}{a} - \frac{1}{b}\right)$$

The potential difference must be the same for all paths between the conductors.

Since $\Delta V_1 = \Delta V_2$, if $K_1 \neq K_2$, $Q_1 \neq Q_2$.

Let $Q_1 = 4\pi a^2 \sigma_1$ $\qquad Q_2 = 4\pi a^2 \sigma_2$

$$\Delta V = \Delta V_1 = \frac{-4\pi a^2 \sigma_1}{4\pi\varepsilon_0 K_1}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= \frac{\sigma_1}{K_1 \varepsilon_0}\left(a - \frac{a^2}{b}\right)$$

$$\sigma_1 = \frac{\kappa_1 \varepsilon_0 |\Delta V|}{\left(a - a^2/b\right)}$$

$$\sigma_2 = \frac{\kappa_2 \varepsilon_0 |\Delta V|}{\left(a - a^2/b\right)}$$

Note - These fields satisfy electrostatic boundary conditions. $\vec{E}$ is the same in both regions and tangent to the interface.

$$C = \frac{Q}{|\Delta V|} = \frac{2\pi a^2 \sigma_1 + 2\pi a^2 \sigma_2}{|\Delta V|}$$

$$= 2\pi a^2 \left(\frac{\kappa_1 \varepsilon_0}{\left(a - a^2/b\right)} + \frac{\kappa_2 \varepsilon_0}{\left(a - a^2/b\right)} \right)$$

$$= \frac{2\pi \varepsilon_0 \left(\kappa_1 + \kappa_2\right)}{\frac{1}{a} - \frac{1}{b}}$$