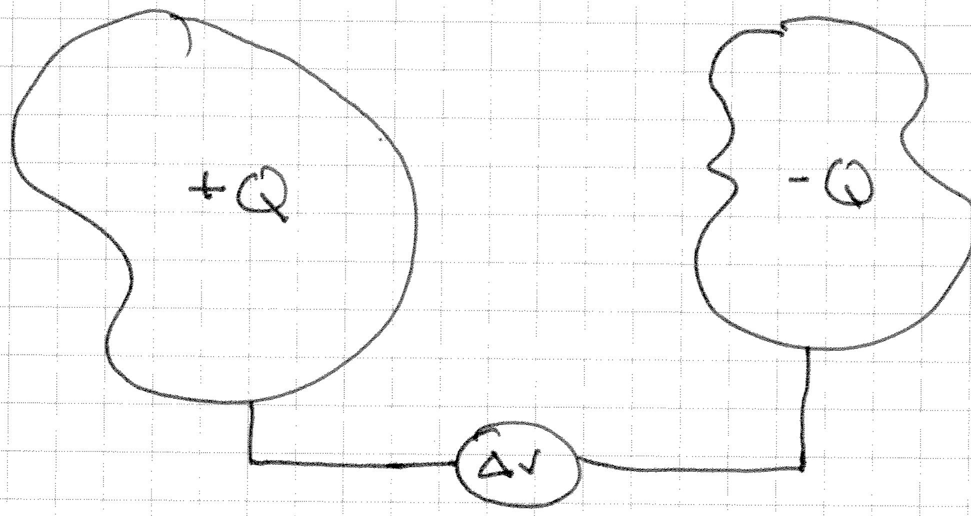
Capacitance

Consider two conductors. If $+Q$ is added to one and $-Q$ to the other, there will be a difference in potential of ΔV between the two conductors.



If $\pm Q$ is increased, ΔV increases proportionally since V is linear in Q, therefore the ratio $Q/\Delta V$ is constant.

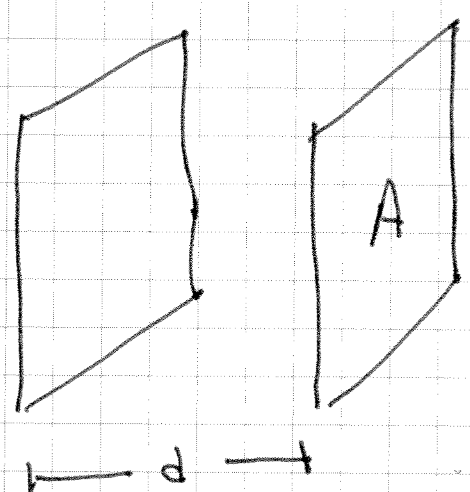
Dfn Capacitance

$$C = \frac{Q}{|\Delta V|}$$

Properties of Capacitance

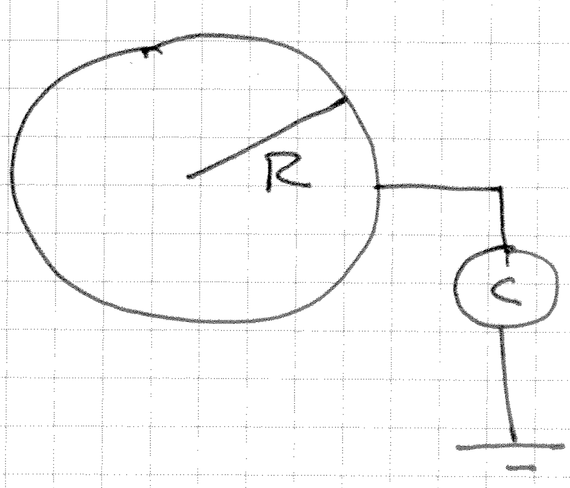
- Depends only on geometry, not Q, ΔV.
- Unit Farad $1\,F = 1\,C/V$
- $\varepsilon_0 = 8.85 \times 10^{-12}\ F/m$

Parallel-Plate Capacitor



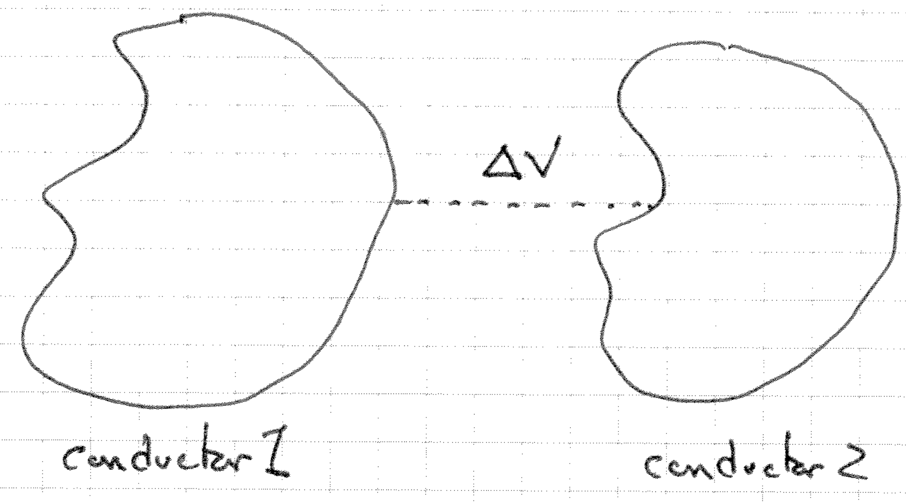
$$C = \frac{\varepsilon_0 A}{d}$$

Isolated Spherical Capacitor (other plate is ground)



$$C = 4\pi\varepsilon_0 R$$

Energy Stored in Capacitor

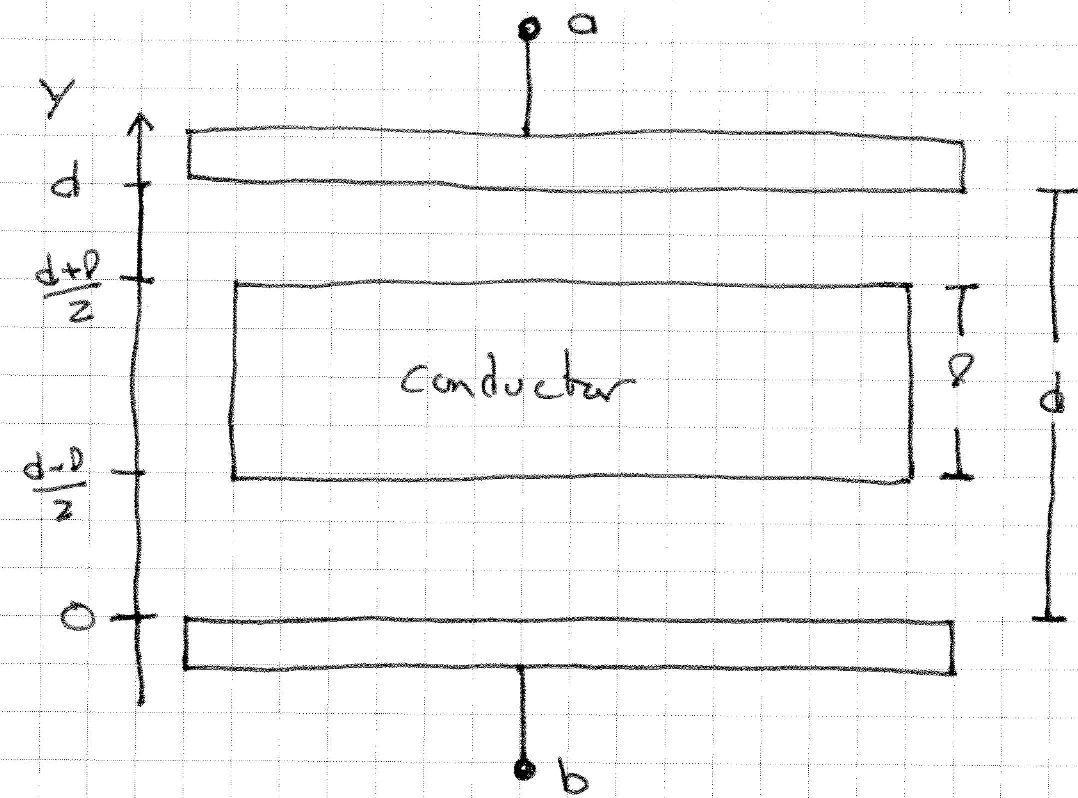


Build charge $\pm Q$, dQ at a time. The work to move charge dQ from conductor 1 to conductor 2 is

$$dW = dQ\,\Delta V$$
$$= dQ\,\frac{Q}{C}$$

$$\text{Total Work} = \text{Energy} = U = \int_0^Q dW = \int_0^Q dQ\,\frac{Q}{C}$$
$$= \frac{1}{2}\frac{Q^2}{C}$$
$$= \frac{1}{2} Q\Delta V$$
$$= \frac{1}{2} C(\Delta V)^2$$

Ex Compute capacitance of two parallel plates with area A and separation d. The plates contain a conducting spacer of width ℓ.



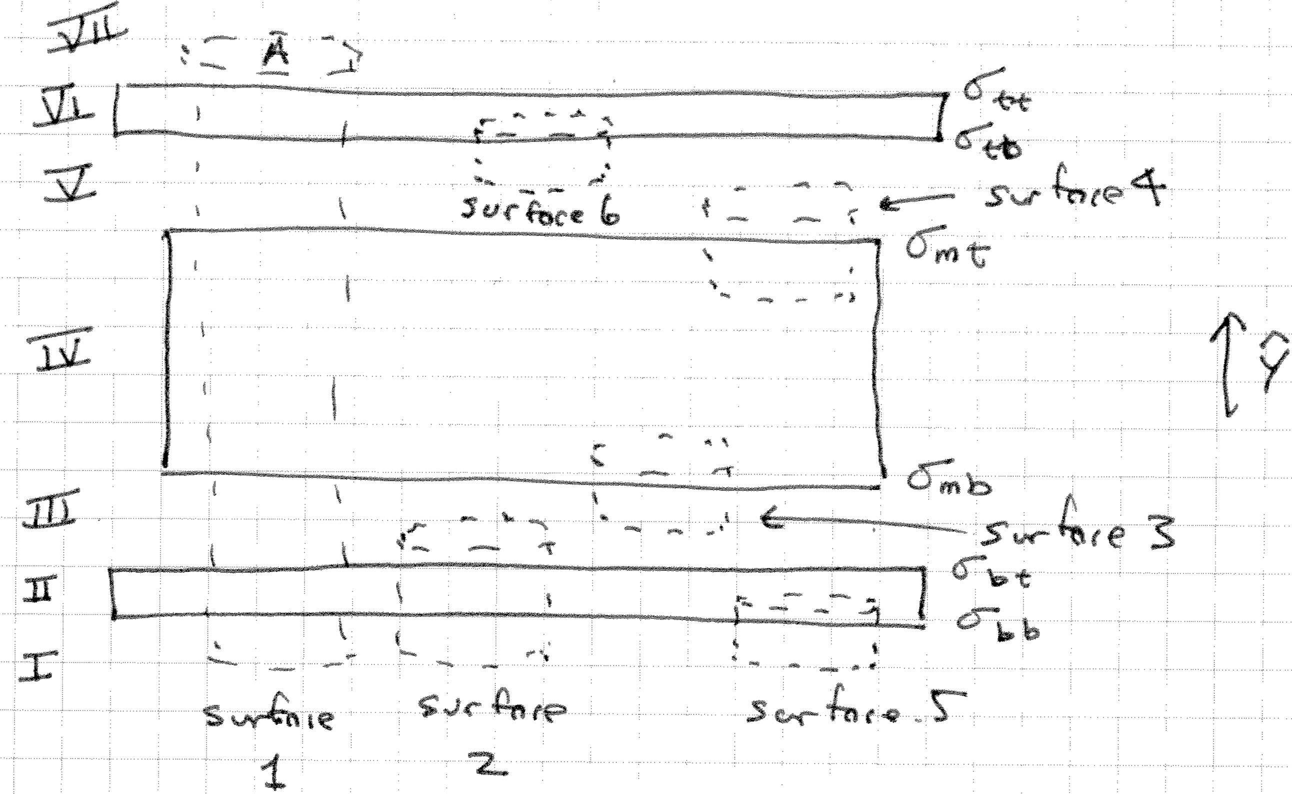
I. Place $+Q$ on bottom plate, $-Q$ on top plate.

This produces a total charge density of

$$\sigma_b = \frac{Q}{A_p} \equiv \sigma \qquad \sigma_a = \frac{-Q}{A_p} \equiv -\sigma$$

where A_p is the area of the plate.

II. Use Gauss' Law to Find the Fields



The Gaussian surfaces we will need are drawn above.

<u>Surface 1</u> $\quad Q_{enc} = \sigma_a A + \sigma_b A = 0$

Let $\vec{E}_i = E_i \hat{y}$

$$\Phi_e = E_{VII} A - E_I A = \frac{Q_{enc}}{\varepsilon_0} = 0 \quad \text{Gauss}$$

<u>Symmetry</u>

$$E_{VII} = -E_I$$

$$2E_{VII} = 0$$

$$\vec{E}_{VII} = 0 \qquad \vec{E}_I = 0$$

Surface 5 Let the charge density on the bottom face of the bottom conductor be σ_{bb}, the charge density on the top face of the bottom conductor be σ_{bt} etc.

$$\Phi_e = \underset{\substack{\| \\ 0 \\ \text{conductor}}}{E_{II}} A - \underset{\substack{\| \\ 0}}{E_{I}} A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma_{bb} A}{\varepsilon_0}$$

$$\Rightarrow \sigma_{bb} = 0 \qquad \Rightarrow \sigma_{bt} = \sigma$$

Surface 2 $\quad Q_{enc} = \sigma A$

$$\Phi_e = E_{III} A - \underset{\substack{\| \\ 0}}{E_{I}} A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\vec{E}_{III} = \frac{\sigma}{\varepsilon_0} \hat{y}$$

Surface 3 $\quad Q_{enc} = \sigma_{mb} A$

$$\vec{E}_{IV} = 0 \quad \text{conductor}$$

$$\underset{\substack{\| \\ 0}}{E_{IV}} A - E_{III} A = Q_{enc}/\varepsilon_0 = \sigma_{mb} A / \varepsilon_0$$

$$\sigma_{mb} = -E_{III}\,\varepsilon_0 = -\sigma$$

The central conductor is neutral so

$$\sigma_{mb} + \sigma_{mt} = 0$$

$$\sigma_{mt} = +\sigma$$

<u>Surface 4</u> $\quad Q_{enc} = \sigma_{mt} A = \sigma A$

$$\underset{\underset{0}{\shortparallel}}{E_{V} A} - E_{IV} A = \frac{Q_{enc}}{\varepsilon_0} = \sigma A$$

$$\vec{E}_V = \frac{\sigma}{\varepsilon_0} \hat{y}$$

<u>Surface 6</u> $\quad Q_{enc} = \sigma_{tb} A$

$$E_{VI} A - E_V A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma_{tb} A}{\varepsilon_0}$$

$$\vec{E}_{VI} = 0 \quad \text{(conductor)}$$

$$\sigma_{tb} = -\varepsilon_0 E_V = -\sigma$$

The total charge density on the top plate is $-\sigma$

$$-\sigma = \sigma_{tb} + \sigma_{tt}$$

$$\Rightarrow \sigma_{tt} = 0$$

Summary

$\vec{E}_{I} = 0$

$\vec{E}_{II} = 0$

$\vec{E}_{III} = \frac{\sigma}{\varepsilon_0}\hat{y}$

$\vec{E}_{IV} = 0$

$\vec{E}_{V} = \frac{\sigma}{\varepsilon_0}\hat{y}$

$\vec{E}_{VI} = 0$

$\vec{E}_{VII} = 0$

$\sigma_{bb} = 0$

$\sigma_{bt} = \sigma$

$\sigma_{mb} = -\sigma$

$\sigma_{mt} = \sigma$

$\sigma_{tb} = -\sigma$

$\sigma_{tt} = 0$

Potential Difference ΔV_{ba}

$$\Delta V_{ba} = -\int_{b \to a} \vec{E} \cdot d\vec{l}$$

$$= \Delta V_{III} + \Delta V_{V}$$

$$\Delta V_{III} = -\int_0^{(d-\ell)/2} \vec{E}_{III} \cdot \hat{y}\, dy \qquad d\ell = \hat{y}\, dy$$

$$= -\int_0^{(d-\ell)/2} \frac{\sigma}{\varepsilon_0} \hat{y} \cdot \hat{y}\, dy$$

$$= -\frac{\sigma}{\varepsilon_0} \int_0^{(d-\ell)/2} dy$$

$$= -\frac{\sigma}{\varepsilon_0} \left(\frac{d-\ell}{2} \right)$$

$$= -E \Delta y$$

<u>Check Sign</u> Potential decreases along field line

$$\Longrightarrow - \checkmark$$

$$\Delta V_V = -\frac{\sigma}{\varepsilon_0} \left(\frac{d-\ell}{2} \right)$$

$$\Delta V_{ba} = -\frac{\sigma}{\varepsilon_0} (d-\ell)$$

<u>Dfn Capacitance</u>

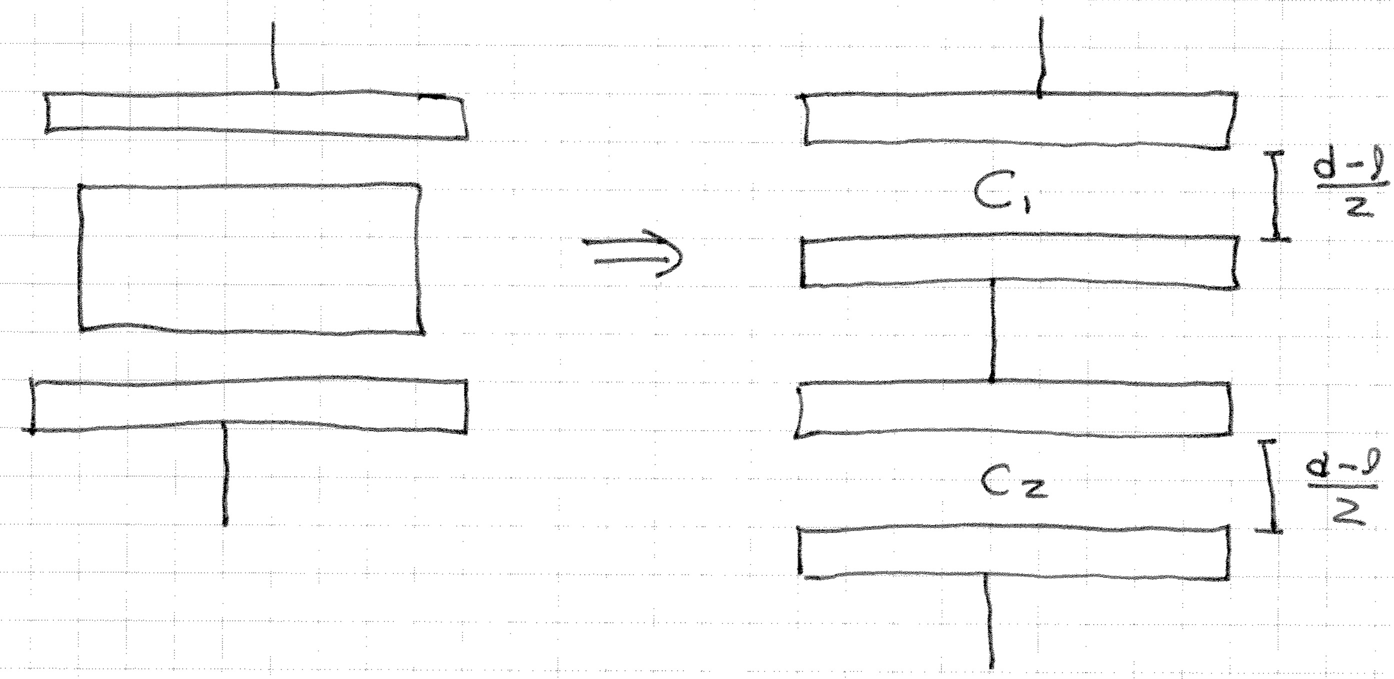
$$C = \frac{Q}{|\Delta V|} = \frac{\sigma A}{|\Delta V|} = \frac{\sigma A}{\frac{\sigma}{\varepsilon_0}(d-\ell)} = \frac{A \varepsilon_0}{(d-\ell)}$$

Capacitors Add in Series As

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

Capacitors Add in Parallel As

$$C_{eq} = C_1 + C_2$$

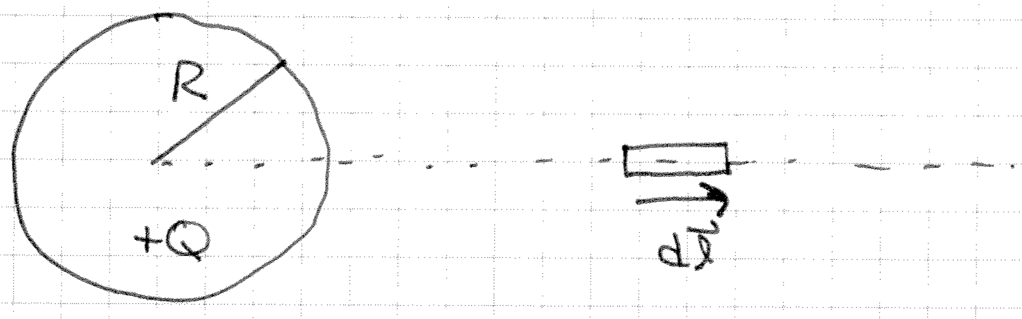


$$C_1 = \frac{\varepsilon_0 A}{(d-\ell)/2} \qquad C_2 = \frac{\varepsilon_0 A}{(d-\ell)/2} = \frac{2\varepsilon_0 A}{d-\ell}$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d-\ell}{2\varepsilon_0 A} + \frac{d-\ell}{2\varepsilon_0 A}$$

$$C_{eq} = \frac{\varepsilon_0 A}{(d-\ell)}$$

Ex Capacitance Isolated Sphere + Ground



I. Add +Q to Sphere

II Field $r > R$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{r}$$

III Potential Difference $\Delta V_{R\infty}$

$$\Delta V_{R\infty} = -\int_{R\to\infty} \vec{E}\cdot d\vec{l} \qquad d\vec{l} = \hat{r}\,dr$$

$$= -\int_R^\infty \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{r}\cdot\hat{r}\,dr$$

$$= \frac{-Q}{4\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2}$$

$$\Delta V_{R\infty} = \frac{-Q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_R^{\infty}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\infty} - \frac{1}{R}\right) = -\frac{Q}{4\pi\varepsilon_0 R}$$

<u>Dfn Capacitance</u>

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{Q/4\pi\varepsilon_0 R}$$

$$= 4\pi\varepsilon_0 R$$

<u>Ex</u> Energy Stored in thin spherical shell with charge density σ and radius R.

$\Longrightarrow$ Energy must be positive because an external agent must do work to assemble system.

Method I - The system has the same charge distribution as the isolated spherical capacitor.

$$U = \frac{1}{2}\frac{Q^2}{C}$$

$$= \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 R} \qquad Q = 4\pi R^2 \sigma$$

$$U = \frac{1}{2}\frac{(4\pi R^2\sigma)^2}{4\pi\varepsilon_0 R}$$

$$= \frac{2\pi R^3 \sigma^2}{\varepsilon_0}$$

Method II All charge on the shell is at the same potential $V = \frac{Q}{4\pi\varepsilon_0 R} = \frac{Q}{4\pi\varepsilon_0 R}$

$$U = \frac{1}{2}\int dq\, V = \frac{1}{2}\int_{\text{sphere}} \sigma\, da\, V$$

$$= \frac{1}{2}\sigma V \int_{\text{sphere}} da = 4\pi R^2 \sigma V = \frac{QV}{2}$$

$$= \frac{Q^2}{8\pi\varepsilon_0 R} = \frac{2\pi R^3\sigma^2}{\varepsilon_0}$$

Method III   Energy density

$$u = \frac{1}{2}\varepsilon_0 E^2 \qquad \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$= \frac{1}{2}\varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2$$

$$= \frac{Q^2}{32\pi^2\varepsilon_0 r^4}$$

Total Energy

$$U = \int_{space} u\, d\tau \qquad d\tau = 4\pi r^2 dr$$

$$= \int_R^\infty 4\pi r^2 u\, dr = \int_R^\infty 4\pi r^2\left(\frac{Q^2}{32\pi^2\varepsilon_0 r^4}\right)dr$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2}$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\frac{1}{R} \quad ✓$$