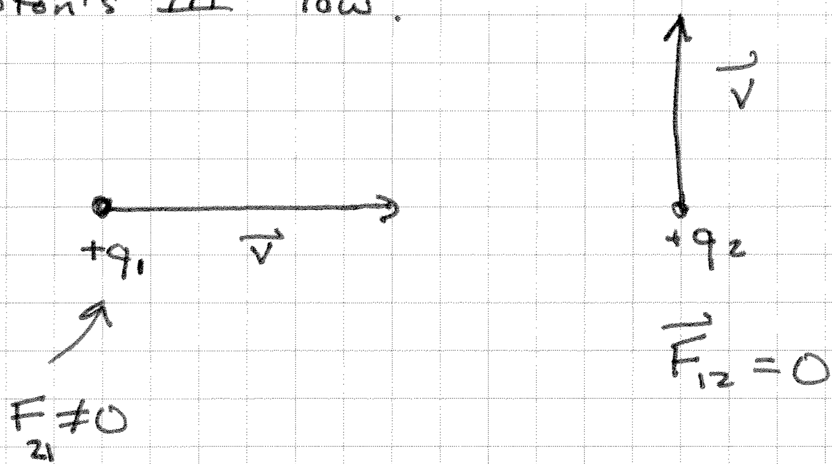
Conservation of Momentum

It is fairly easy to find situations where the electromagnetic force appears not to obey Newton's III law.



but Newton's III law is just a statement of the parwise conservation of momentum

$$\vec{F}_{12} = -\vec{F}_{21}$$

$$\frac{d\vec{p}_{12}}{dt} = -\frac{d\vec{p}_{21}}{dt}$$

The electromagnetic field must conserve momentum or the universe is in trouble.

$\Longrightarrow$ We would like to write a continuity eqn for momentum.

To do this we need to express a flow of momentum. Suppose we lived in one dimension and T is the momentum flux.

$$T = \frac{\text{momentum}}{\text{area} \cdot \text{time}}$$

then

Momentum flowing through the surface surrounding the volume.

$$\frac{dp}{dt} = \int_S T\,da = F$$

The change in momentum of some volume

$\Rightarrow$ The force is the integral of the momentum flux over the surface of the object.

$\Rightarrow$ The problem is that we don't live in one dimension so both momentum and flux are vectors, so we need things like the flux of $\hat{x}$ directed momentum in the $\hat{y}$ direction.

Therefore our flux T_{ij} must have two components

$$T_{xx} = \frac{x\text{-momentum}}{x\text{-Area} \cdot \text{Time}}$$

$$T_{yx} = \frac{y\text{-momentum}}{x\text{-Area} \cdot \text{Time}}$$

$\Rightarrow$ To get the momentum per unit time flowing through a surface multiple T by the area.

$\Rightarrow$ If T constant, to get the z-momentum flowing through the surface $A_y \hat{y}$ in the x-z plane, we would calculate

$$F_z = \frac{dp_z}{dt} = T_{zy} A_y$$

If we wanted the other components

$$F_x = \frac{dp_x}{dt} = T_{xy} A_y$$

$$F_y = \frac{dp_y}{dt} = T_{yy} A_y$$

The notation looks like matrix multiplication.

$$\vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \frac{d\vec{p}}{dt} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ A_y \\ 0 \end{pmatrix}$$

Or more generally for the arbitrary surface

$$\vec{A} = A\hat{n} = A(n_x, n_y, n_z)$$

$$\vec{F} = \frac{d\vec{p}}{dt} = \underbrace{\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}}_{\equiv \overleftrightarrow{T}} \underbrace{\begin{pmatrix} An_x \\ An_y \\ An_z \end{pmatrix}}_{= \vec{A}}$$

Stress Tensor $(\overleftrightarrow{T})$

$$\vec{F} = \frac{d\vec{p}}{dt} = \overleftrightarrow{T} \cdot \vec{A}$$

$\Rightarrow$ Note, the above dot product accomplished the matrix multiplication and yields a vector.

Additional notation

Let $\vec{T}_x$ be the flux of x-momentum

$$\vec{T}_x = (T_{xx}, T_{xy}, T_{xz})$$

If all goes well we will write a continuity equation using $\vec{T}_i$ and the momentum densities $\vec{\wp}_i$ such as

$$\frac{\partial \wp_x}{\partial t} + \nabla \cdot \vec{T}_x = 0$$

$$\frac{\partial \wp_y}{\partial t} + \nabla \cdot \vec{T}_y = 0$$

$$\frac{\partial \wp_z}{\partial t} + \nabla \cdot \vec{T}_z = 0$$

— or —

$$\frac{\partial \vec{\wp}}{\partial t} + \nabla \cdot \overleftrightarrow{T} = 0$$

$\Rightarrow$ Note, we will miss this goal by a $-$ sign because of the convention for defining $\overleftrightarrow{T}$.

$$\nabla \cdot \overleftrightarrow{T} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

Solve for $\overleftrightarrow{T}$

The total momentum of an isolated system can separated into the momentum of its particles $\vec{p}_{mech}$ and any momentum associated with its fields, $\vec{p}_{em}$

$$\vec{p} = \vec{p}_{mech} + \vec{p}_{em}$$

$\vec{p}_{mech}$ changes because the fields exert forces on the particles.

Let ρ be the charge density of the system. The force per unit volume $\vec{f}$ exerted by the fields is

$$\vec{f} = \rho \left(\vec{E} + \vec{v} \times \vec{B} \right) = \rho \vec{E} + \vec{J} \times \vec{B}$$

The total force on the particles and therefore the time rate of change of mechanical momentum is:

$$\vec{F} = \int_V \vec{f}\, d\tau = \frac{d\vec{p}_{mech}}{dt}$$

$$= \int_V \left(\rho \vec{E} + \vec{J} \times \vec{B} \right) d\tau$$

Eliminate $\rho, \vec{J}$ using Maxwell's eqns

$$\rho = \varepsilon_0\, \nabla \cdot \vec{E}$$

$$\vec{J} = \frac{\nabla \times \vec{B}}{\mu_0} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

So

$$\rho \vec{E} + \vec{J} \times \vec{B} = \varepsilon_0\, \vec{E} \left(\nabla \cdot \vec{E} \right) + \frac{1}{\mu_0} \left(\nabla \times \vec{B} \right) \times \vec{B}$$

$$- \varepsilon_0 \left(\frac{\partial \vec{E}}{\partial t} \right) \times \vec{B}$$

Work on this term by term

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \left(\frac{\partial \vec{E}}{\partial t}\right)\times\vec{B} + \vec{E}\times\frac{\partial \vec{B}}{\partial t}$$

Use Faraday's Law $\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\Rightarrow \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \left(\frac{\partial \vec{E}}{\partial t}\right)\times\vec{B} - \vec{E}\times(\nabla\times\vec{E})$$

$$\left(\frac{\partial \vec{E}}{\partial t}\right)\times\vec{B} = \vec{E}\times(\nabla\times\vec{E}) + \frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

So

$$\vec{f} = \rho\vec{E} + \vec{J}\times\vec{B} = \varepsilon_0(\nabla\cdot\vec{E})\vec{E} - \frac{1}{\mu_0}\vec{B}\times(\nabla\times\vec{B})$$

$$- \varepsilon_0\vec{E}\times(\nabla\times\vec{E}) - \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

where I used $(\nabla\times\vec{B})\times\vec{B} = -\vec{B}\times(\nabla\times\vec{B})$

Vector Identity (front cover)

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A}$$

Let $\vec{A}, \vec{B} \rightarrow \vec{E}$

$$\nabla(E^2) = \vec{E}\times(\nabla\times\vec{E}) + \vec{E}\times(\nabla\times\vec{E}) + (\vec{E}\cdot\nabla)\vec{E} + (\vec{E}\cdot\nabla)\vec{E}$$

$$= 2\vec{E}\times(\nabla\times\vec{E}) + 2(\vec{E}\cdot\nabla)\vec{E}$$

$$\Longrightarrow \vec{E}\times(\nabla\times\vec{E}) = \frac{1}{2}\nabla(E^2) - (\vec{E}\cdot\nabla)\vec{E}$$

$\Longrightarrow$ Use the same trick on B

$$\vec{f} = \varepsilon_0(\nabla\cdot\vec{E})\vec{E} - \frac{1}{2}\varepsilon_0\nabla(E^2) + \varepsilon_0(\vec{E}\cdot\nabla)\vec{E}$$

$$-\frac{1}{2\mu_0}\nabla(B^2) + \frac{1}{\mu_0}(\vec{B}\cdot\nabla)\vec{B}$$

$$-\varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

Collect similar terms

$$\vec{f} = \varepsilon_0\Big((\nabla\cdot\vec{E})\vec{E} + (\vec{E}\cdot\nabla)\vec{E} \Big)$$

$$+\frac{1}{\mu_0}\Big((\nabla\cdot\vec{B})\vec{B} + (\vec{B}\cdot\nabla)\vec{B} \Big)$$

$$-\frac{1}{2}\nabla\Big(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \Big)$$

$\nabla\cdot$ Stress Tensor

$$-\varepsilon_0 \frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)$$

$\Rightarrow$ I added an extra $\nabla\cdot\vec{B}=0$

$\Rightarrow$ The third term is the gradient of the energy density.

$\Rightarrow$ The fourth term is related to the time rate of change of the energy flux.

$\Rightarrow$ ~~The~~ first three terms are $\nabla\cdot\overleftrightarrow{T}$.

Maxwell Stress Tensor ($\overleftrightarrow{T}$)

The first three terms in the previous expression. If you unpack everything $\overleftrightarrow{T}$ becomes. The first three terms are $\nabla \cdot \overleftrightarrow{T}$

$$T_{ij} = \varepsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right)$$

$$+ \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)$$

Kronecker Delta

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Example

$$T_{xx} = \varepsilon_0 \left(E_x^2 - \frac{1}{2} E^2 \right) + \frac{1}{\mu_0} \left(B_x^2 - \frac{1}{2} B^2 \right)$$

$$= \varepsilon_0 \left(\frac{1}{2} E_x^2 - \frac{1}{2} E_y^2 - \frac{1}{2} E_z^2 \right)$$

$$+ \frac{1}{\mu_0} \left(\frac{1}{2} B_x^2 - \frac{1}{2} B_y^2 - \frac{1}{2} B_z^2 \right)$$

or $\quad T_{xy} = \varepsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y$

In terms of $\overleftrightarrow{T}$ the force density becomes

$$\vec{f} = \nabla \cdot \overleftrightarrow{T} - \varepsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$$

The total force on a volume V is

$$\vec{F} = \int_V \vec{f}\, d\tau = \int_V \nabla \cdot \overleftrightarrow{T} d\tau - \varepsilon_0 \mu_0 \frac{d}{dt} \int_V \vec{S}\, d\tau$$

$$= \frac{d\vec{p}_{mech}}{dt} = -\oint_S \overleftrightarrow{\wp} \cdot d\vec{a} - \frac{d}{dt} \vec{p}_{em}$$

<u>Momentum Stored in EM fields ($\vec{p}_{em}$)</u>

$$\vec{p}_{em} = \int_V \mu_0 \varepsilon_0 \vec{S}\, d\tau$$

<u>Momentum Density of the Fields $\vec{\wp}_{em}$</u>

$$\vec{\wp}_{em} = \mu_0 \varepsilon_0 \vec{S}$$

Momentum Flux into V ($\overleftrightarrow{\wp}$)

$$\overleftrightarrow{\wp} = \overleftrightarrow{T}$$

$$\frac{d\vec{p}_{mech}}{dt} = -\frac{d\vec{p}_{em}}{dt} + \oint_S \overleftrightarrow{T} \cdot d\vec{a}$$

=	=	=
change of mechanical momentum of system	change of momentum of fields	momentum flowing into system across S

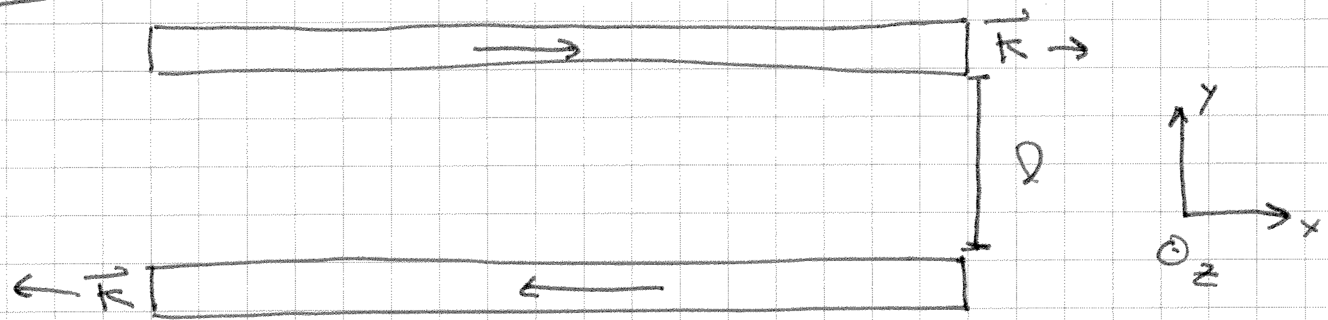
Let $\vec{\wp}_{mech}$ be density of mechanical momentum, so the continuity equation for momentum becomes

$$\frac{\partial}{\partial t}\left(\vec{\wp}_{mech} + \vec{\wp}_{em}\right) = \nabla \cdot \overleftrightarrow{T}$$

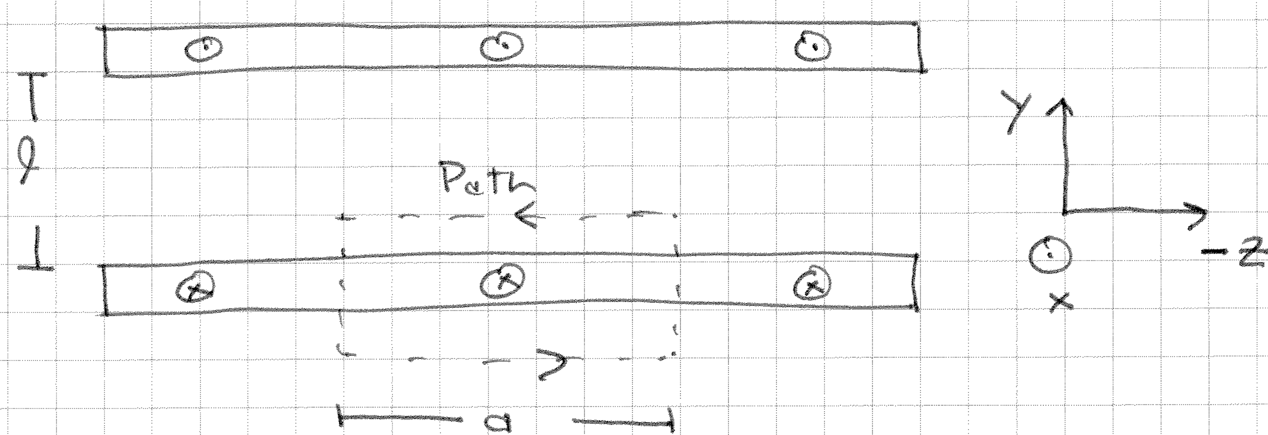
(note unexpected − sign)

Ex Compute momentum per unit area flowing between two current sheets carrying current $\vec{K}$ held at potential difference ΔV.

Side View



End View



$\Longrightarrow$ Normal of path outward by RHR

$\Longrightarrow$ Field zero outside planes

$I_{enc} = -aK$

Ampere $\oint \vec{B}\cdot d\vec{l} = +B_i a = \mu_0 I_{enc} = -aK\mu_0$

$$\vec{B}_i = -\mu_0 K\,\hat{z}$$

Electric Field $\qquad \vec{E} = \frac{\Delta V}{l}\hat{y}$

Poynting Vector

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$$

$$= \frac{1}{\mu_0}\left(\frac{\Delta V}{l}\hat{y}\right)\times\left(-\mu_0 K\hat{z}\right)$$

$$= -\frac{\Delta V}{l}K\hat{x}$$

Momentum Density of Field

$$\vec{\wp}_{em} = +\mu_0\epsilon_0\vec{S}$$

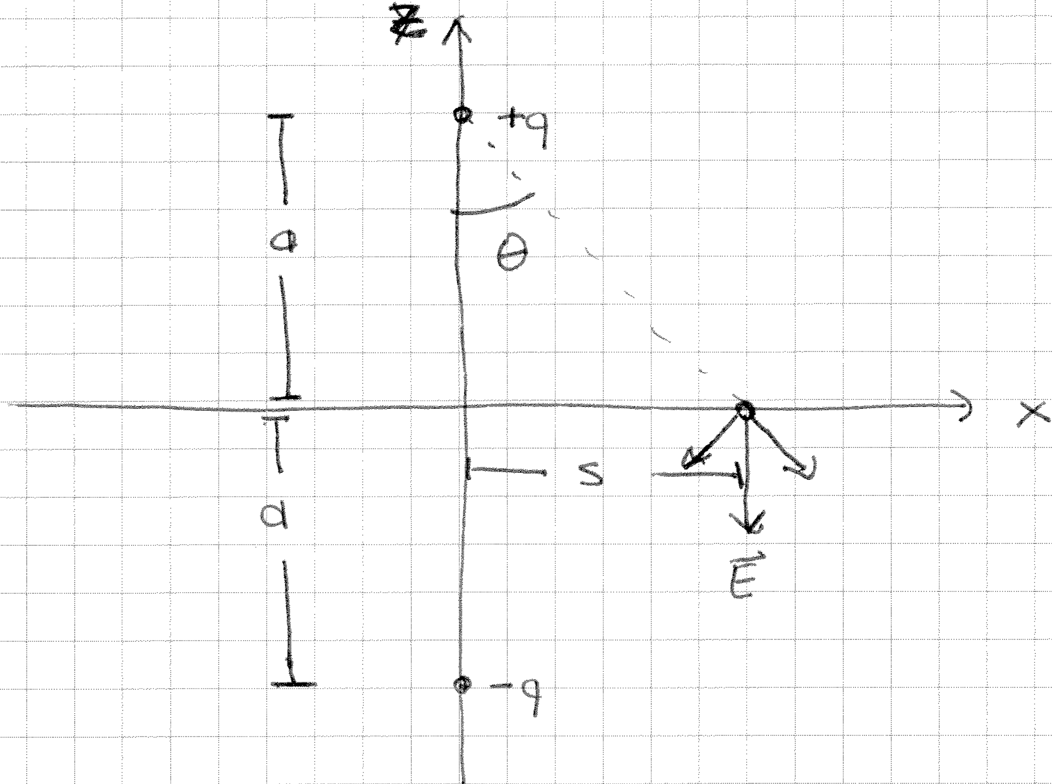
$$= -\mu_0\epsilon_0\frac{\Delta V}{l}K\hat{x}$$

Momentum per unit Area $(\vec{P})$

$$\vec{P} = l\,\vec{\wp}_{em} = -\mu_0\epsilon_0\Delta V K\hat{x}$$

$\Rightarrow$ Note, this momentum is balance by a relativistic momentum associated with the currents.

8.4(b) Compute the force a $-q$ charge exerts on a $+q$ charge using the stress tensor.



As before

$$\vec{E} = \frac{-2kq}{(a^2+s^2)} \underbrace{\frac{a}{\sqrt{a^2+s^2}}}_{\cos\theta} \hat{z}$$

$$= -\frac{qa}{2\pi\epsilon_0 (a^2+s^2)^{3/2}} \hat{z}$$

The force on the $+q$ charge is found by integrating $\overleftrightarrow{T}$ over the bounding surface, the x-y plane, with outward normal $-\hat{z}$.

$$\vec{F} = \int_S \overleftrightarrow{T} \cdot d\vec{a} \qquad d\vec{a} = (-\hat{z})da$$

$$= -\hat{z} \int_S T_{zz}\, dx$$

$$T_{zz} = \varepsilon_0 \left(E_z^2 - \frac{1}{2} E^2 \right) + \frac{1}{\mu_0} \underbrace{\left(B_z^2 - \frac{1}{2} B^2 \right)}_{= 0}$$

$$E^2 = E_z^2 \quad \text{here}$$

$$T_{zz} = \frac{1}{2} \varepsilon_0 E_z^2 = \frac{q^2 a^2}{8 \pi^2 \varepsilon_0 (a^2 + s^2)^3}$$

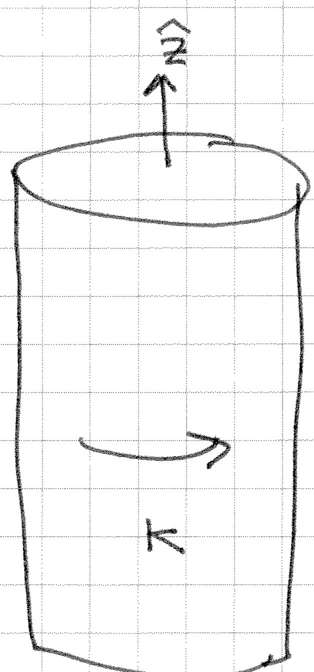
$$\vec{F} = -\hat{z} \int_{\text{plane}} T_{zz}\, da = -\hat{z} \int_0^\infty s\,ds \int_0^{2\pi} d\phi\, T_{zz}$$

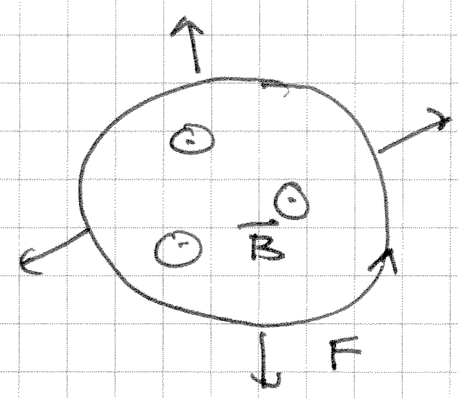
$$\vec{F} = -\frac{\hat{z}\, q^2 a^2}{8\pi^2 \varepsilon_0} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\infty} \frac{s}{(a^2+s^2)^3}\, ds}_{\frac{1}{4a^4}}$$

$$\vec{F} = -\frac{\hat{z}\, q^2}{16\pi\varepsilon_0 a^2} = -\frac{q^2}{4\pi\varepsilon_0 (z_0)^2}\,\hat{z}$$

<u>Ex</u> Force on current in solenoid ~~but~~ due to field

$$\vec{B} = B_0\,\hat{z} = n\mu_0 I\,\hat{z} = \mu_0 K\,\hat{z}$$





Stress Tensor

$$T_{ij} = \underbrace{\varepsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right)}_{=0} + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right)$$

$$T_{xx} = \frac{1}{\mu_0}\left(\underbrace{B_x B_x}_{=0} - \frac{1}{2} B^2 \right) = \frac{-B_0^2}{2\mu_0} = T_{yy}$$

$$T_{zz} = \frac{1}{\mu_0}\left(B_z^2 - \frac{1}{2} B^2 \right) = \frac{B_0^2}{2\mu_0}$$

$$T_{xy} = T_{xz} = T_{yz} = 0$$

$$\overleftrightarrow{T} = \begin{pmatrix} \frac{-B_0^2}{2\mu_0} & 0 & 0 \\ 0 & \frac{-B_0^2}{2\mu_0} & 0 \\ 0 & 0 & \frac{B_0^2}{2\mu_0} \end{pmatrix}$$

$$\vec{F} = \int_S \overleftrightarrow{T} \cdot d\vec{a}$$

The outward normal for the universe outside the solenoid is $-\hat{s}$, $d\vec{a} = -\hat{s}\, da$.

The pressure, force per unit area, when $\hat{s} = \hat{x}$ is $\overleftrightarrow{T} \cdot (-\hat{x}\, da)$

Let f_x be the force per unit area,

$$f_x = \overleftrightarrow{T} \cdot (-\hat{x}) = -T_{xx} = \frac{B_0^2}{2\mu_0}$$

~~Using more traditional methods, the magnetic pressure~~ is

$$\cancel{\frac{d\vec{F}}{da} = \vec{K} \times \vec{B} = KB}$$

Work out the magnetic pressure traditionally.

The force on a ring of current of length L is

$$F = ILB = 2\pi R I B$$

If the current I is a current density K flowing through a strip of width dz, then

$$I = K\,dz$$

$$F = 2\pi R K\,dz\,B$$

The force per unit area is then

$$\frac{F}{2\pi R\,dz} = KB$$

But remember we have to use ~~the~~ the average field

$$\frac{F}{\text{Area}} = \frac{KB}{2} \qquad B = \mu_0 K$$

$$= \frac{B^2}{2\mu_0} \checkmark$$

$\Rightarrow$ The stress tensor solution did not make use of the currents.