Delta Functions

As we proceed, the electric field of a point charge will be important.

$$\vec{E} = \frac{KQ}{r^2}\hat{r}$$

as will its divergence. Try the divergence thm of $\frac{\hat{r}}{r^2}$.

$$\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

in spherical coordinates.

$$\frac{\hat{r}}{r^2} = \underbrace{\frac{1}{r^2}}_{A_r}\hat{r} + \underbrace{0}_{A_\theta}\hat{\theta} + \underbrace{0}_{A_\phi}\hat{\phi}$$

$$\nabla\cdot\frac{\hat{r}}{r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{1}{r^2}\right) = 0$$

Now apply the divergence Thm

$$\int_V \nabla \cdot \vec{A}\, d\tau = \oint_S \vec{A} \cdot d\vec{a}$$

From the above it looks like

$$\int_V \nabla \cdot \frac{\hat{r}}{r^2}\, d\tau = 0$$

Consider a spherical volume of radius R,

$$d\vec{a} = \cancel{R}\ \hat{r}\, da$$

$$\oint_S \frac{\hat{r}}{r^2} \cdot \hat{r}\, da = \frac{1}{R^2} \oint_S da$$

$$= \frac{1}{R^2} \cdot 4\pi R^2 = 4\pi$$

$$\neq 0$$

Therefore $\nabla \cdot \frac{\hat{r}}{r^2}$ is so singular at the origin that it contributes a finite amount to the integral even though it is only non-zero at a point.

We will call such a function a Dirac Delta Function

$$\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi \delta^3(\vec{r})$$

$$\int_V \delta^3(\vec{r})\, d\tau = 1$$

The delta function is a continuous generalization of the Kronecker delta δ_{ij}

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The delta function uses continoous subscripts

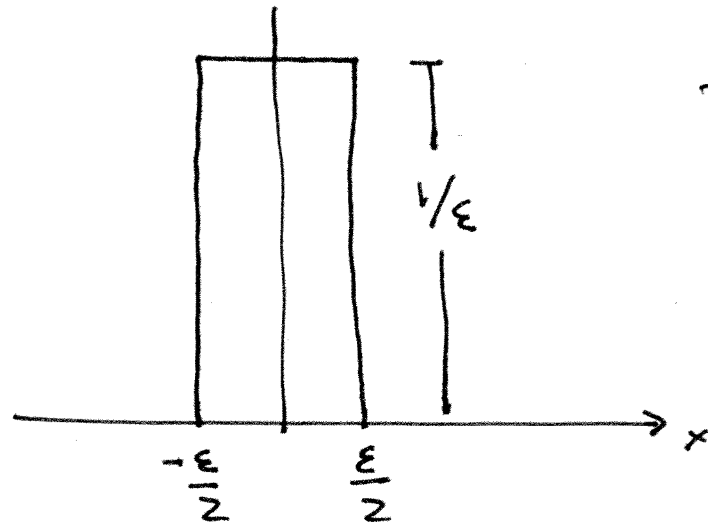
$$\delta_{xx'} = \begin{cases} 0 & \text{if } x \neq x' \\ \infty \text{ (1 in a special way)} & \text{if } x = x' \end{cases}$$

We will write $\delta_{xx'}$ as $\delta(x-x')$ and define $\delta(x-a)$ as a "distribution" s.t.

$$\delta(x-a) = 0 \quad \text{if } x \neq 0 \quad \text{and} \quad \int_{r_1}^{r_2} \delta(x-a)\, dx = 1$$

if $a \in [r_1, r_2]$, 0 otherwise

Note, the delta function is not a real function but is called a "distribution" as the limit of a number of functions.

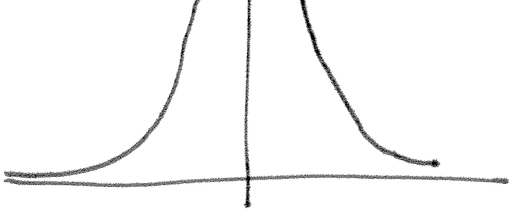


Plot of $\sigma_\varepsilon(x)$

$$\sigma_\varepsilon(x) = \begin{cases} 0 & x < -\varepsilon/2 \\ 1/\varepsilon & -\varepsilon/2 \leq x \leq \varepsilon/2 \\ 0 & x > \varepsilon/2 \end{cases}$$

$$\sigma(x) = \lim_{\varepsilon \to} \sigma_\varepsilon(x)$$

Likewise one can construct a delta function as the limit of a Gaussian



$$\sigma(x) = \lim_{\varepsilon \to 0} \frac{1}{2\sqrt{\pi|\varepsilon|}} e^{\frac{-x^2}{4|\varepsilon|}}$$

Delta functions live under integral signs and make integration easy.

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

Ex

$$\int_{0}^{2\pi} \sin^3(x)\, x^5\, \delta(x-\pi)\,dx$$

$$= \sin^3(\pi)\,\pi^5 = 0$$

To prove something about a delta function, you have to show

$$\int F[\delta(x)]\,f(x)\,dx = \int G[\delta(x)]\,f(x)\,dx$$

for all f. $\Rightarrow$ $F = G$

Ex Show $x \frac{d}{dx} \sigma(x) = -\sigma(x)$

Sln For an arbitrary $f(x)$

$$\int_{-\infty}^{\infty} f(x) \left[x \frac{d\sigma(x)}{dx} \right] dx$$

$$= f(x)\, x\, \sigma(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \sigma(x) \frac{d}{dx} (x f(x))\, dx$$

Integration by parts

The first term is zero since $\sigma(\pm\infty) = 0$.

Work on the second term

$$- \int_{-\infty}^{\infty} \sigma(x) \frac{d}{dx} (x f(x))\, dx$$

$$= - \int_{-\infty}^{\infty} \sigma(x) \left[f(x) + x \frac{df}{dx} \right] dx$$

$$= - \int_{-\infty}^{\infty} \sigma(x) f(x) - \int_{-\infty}^{\infty} \sigma(x)\, x \frac{df}{dx}\, dx$$

The second term is zero since

$$\int_{-\infty}^{\infty} \sigma(x)\, x \frac{df}{dx}\, dx = 0 \left.\frac{df}{dx}\right|_0 = 0$$

So

$$\int_{-\infty}^{\infty} f(x) \left[x \frac{d\sigma}{dx} \right] dx = - \int_{-\infty}^{\infty} \sigma(x) f(x)\, dx$$

for all f

$$\Longrightarrow \quad x \frac{d\sigma}{dx} = -\sigma(x)$$

Three Dimensional Delta Functions

$$\delta^3(\vec{r}-\vec{r}\,') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$\int_V \delta^3(\vec{r}-\vec{r}\,')\,d\tau = 1 \quad \text{if volume contains } \vec{r}\,'$$
$$0 \quad \text{otherwise}$$

$$\int_{\text{space}} \delta^3(\vec{r}-\vec{a})\,f(\vec{r})\,d\tau = f(\vec{a})$$

Curvilinear Coordinates

It is not the case that

$$\delta^3(\vec{r}-\vec{r}) = \delta(s-s')\,\delta(\phi-\phi')\,\delta(z-z')$$

in cylindrical coordinates

In curvilinear coordinates, the volume element can be written in general as

$$d\tau = (h_1\,dq_1)(h_2\,dq_2)(h_3\,dq_3)$$

where q_i is the coordinate.

In cylindrical,

$$d\tau = (ds)(s\,d\phi)(dz)$$

$$q_1 = s \qquad q_2 = \phi \qquad q_3 = z$$

$$h_1 = 1 \qquad h_2 = s \qquad h_3 = 1$$

For points where q_i is uniquely defined

$$\delta^3(\vec{r} - \vec{r}') = \frac{1}{h_1 h_2 h_3}\, \delta(q_1 - q_1')\,\delta(q_2 - q_2')\,\delta(q_3 - q_3')$$

$$= \frac{1}{s}\,\delta(s - s')\,\delta(\phi - \phi')\,\delta(z - z')$$

cylindrical

$$= \frac{1}{r^2 \sin\theta}\,\delta(r - r')\,\delta(\theta - \theta')\,\delta(\phi - \phi')$$

spherical

At some points, multiple q_i may map to the same point.

For example, in cylindrical at $s=0$ all ϕ represent the same point.

At these points, the multiply defined coordinate is dropped and integrated over

$$\delta^3(\vec{r}-\vec{r}\,') = \frac{1}{s\int_0^{2\pi} d\phi}\,\delta(s)\,\delta(z-z')$$

$$= \frac{1}{2\pi s}\,\delta(s)\,\delta(z-z')$$

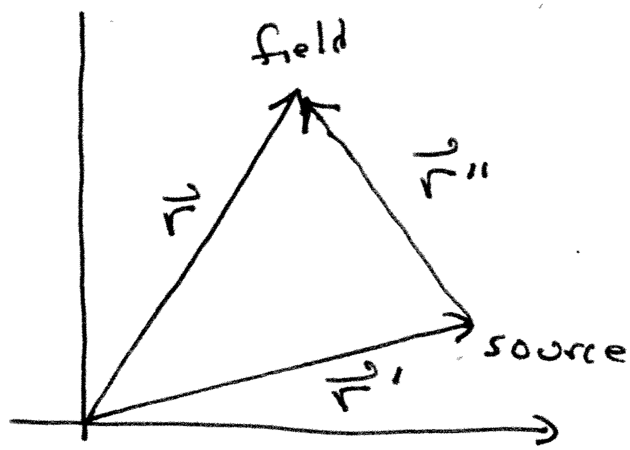
If there are two multiply defined coordinates, for example ϕ, θ at $r=0$, both are integrated.

$$\delta^3(\vec{r}-\vec{r}\,') = \frac{1}{4\pi r^2}\,\delta(r)$$

$$= \frac{1}{r^2\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\theta}\,\delta(r)$$

Displacement and Delta Functions

We will often be interested in the displacement vector $\vec{r}''$ between two points $\vec{r}$ and $\vec{r}'$:



$$\vec{r}'' = \mathscr{r} = \vec{r} - \vec{r}'$$

$$\nabla \cdot \frac{\hat{r}''}{r''^2} = 4\pi\, \delta^3(\vec{r}'')$$

$$\nabla\left(\frac{1}{r''}\right) = -\frac{\hat{r}''}{r''^2}$$

$$\nabla^2\left(\frac{1}{r''}\right) = -4\pi\,\delta^3(\vec{r}'')$$