Electromagnetic Waves

Maxwell's Equations in Vacuum $(\rho = 0, \vec{J} = 0)$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Try to separate the equations, not this will produce equations that are not as general as Maxwell's eqns.

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \qquad \text{(Eqn 11 front cover)}$$

$$= -\nabla \times \frac{\partial \vec{B}}{\partial t} \qquad \text{Faraday}$$

$$= -\frac{\partial}{\partial t}\left(\nabla \times \vec{B}\right)$$

$$= -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \qquad \text{Ampere}$$

Since $\nabla \cdot \vec{E} = 0$,

$$-\nabla^2 E = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Likewise,

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = -\nabla^2 \vec{B}$$

$$= \mu_0 \varepsilon_0 \nabla \times \left(\frac{\partial \vec{E}}{\partial t}\right) \qquad \text{Ampere}$$

$$= \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \times \vec{E})$$

$$= -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \qquad \text{Faraday}$$

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\nabla^2 \vec{B} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

or for example the x-component of $\vec{E}$,

$$\frac{\partial^2 E_x}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0$$

This is an example of the <u>wave equation</u> along with the simple harmonic oscillator one of the most common models of physical systems in the universe.

Note, the two wave equations are not actually independant. The solutions are still related by Maxwell's eqns.

Wave Equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

This equation has general solutions of the form

$$f = g(x-vt) \quad \text{and} \quad f = h(x+vt)$$

where f, g, h are <u>any</u> appropriately differentiable functions.

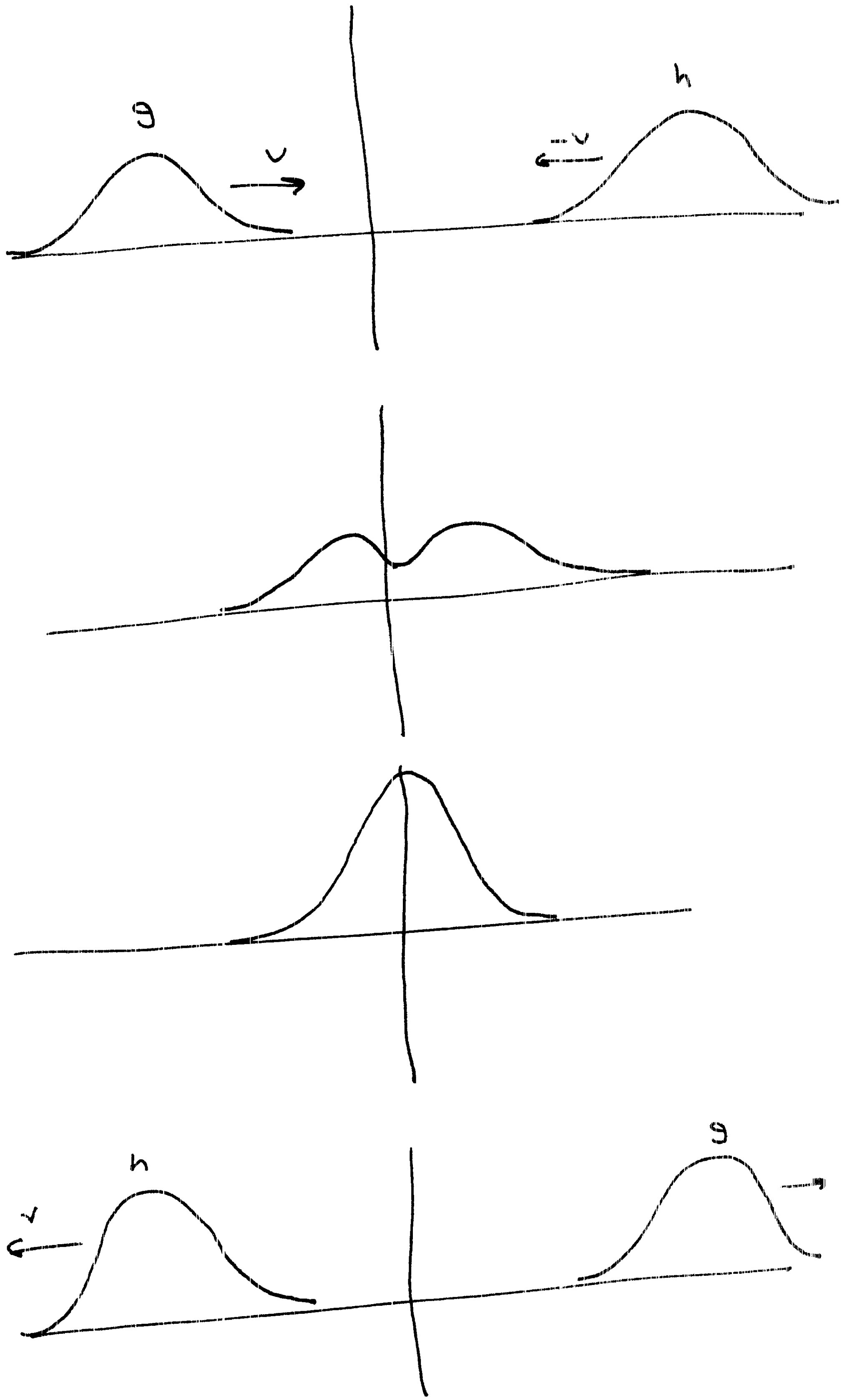
This describes a profile $g(x)$ that travels in the x-direction maintaining its shape of velocity v.

Likewise, $h(x)$ travels in the $-x$ direction with velocity $-v$.

Since the equation is <u>linear</u>, if we have two solutions $g_1(x-vt)$ and $g_2(x-vt)$, ~~thy~~ any linear combination is also a solution

$$f = \alpha g_1 + \beta g_2$$

This leads to some unusual behavior. If we take two pulses travelling in opposite directions



The two pulse pass through each other unchanged.

Notes -

(1) The velocity is fixed by the physical system being considered.

(2) To determine boundary conditions between two systems with different v_i we have to go back to the physics of the systems.

⇒ For vibrating strings, we go back to Newtons laws.

⇒ For EM waves, we go back to Maxwells eqns.

Comparing the wave equation with the equations derived from Maxwell's eqns gives the wave velocity of electromagnetic waves as

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Velocity of Electromagnetic Wave -

$$c \equiv \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 299,792,458 \text{ m/s}$$

$$= 3.0 \times 10^8 \text{ m/s}$$

but this is the speed of light.

Light is an electromagnetic wave!

The most useful solutions to the wave equation are sines and cosines.

$$f(x,t) = \underbrace{A}_{\text{amplitude}} \cos(kx - \omega t + \sigma) \quad \text{right travelling}$$

$$= A \cos(kx + \omega t - \sigma) \quad \text{left travelling}$$

Phase, Phase Constant, Phase Shift (σ) = The wave is delayed by a distance σ / k from reaching the origin at $t=0$.

Wavelength (λ) - Distance for one oscillation

Wavenumber (k) - $k = \frac{2\pi}{\lambda}$

Period (T) - ~~Distance for one osc~~ Time for one oscillation.

Frequency (f) - Oscillations per second

$$f = \frac{1}{T}$$

Angular Frequency (ω) = $\omega = 2\pi f$