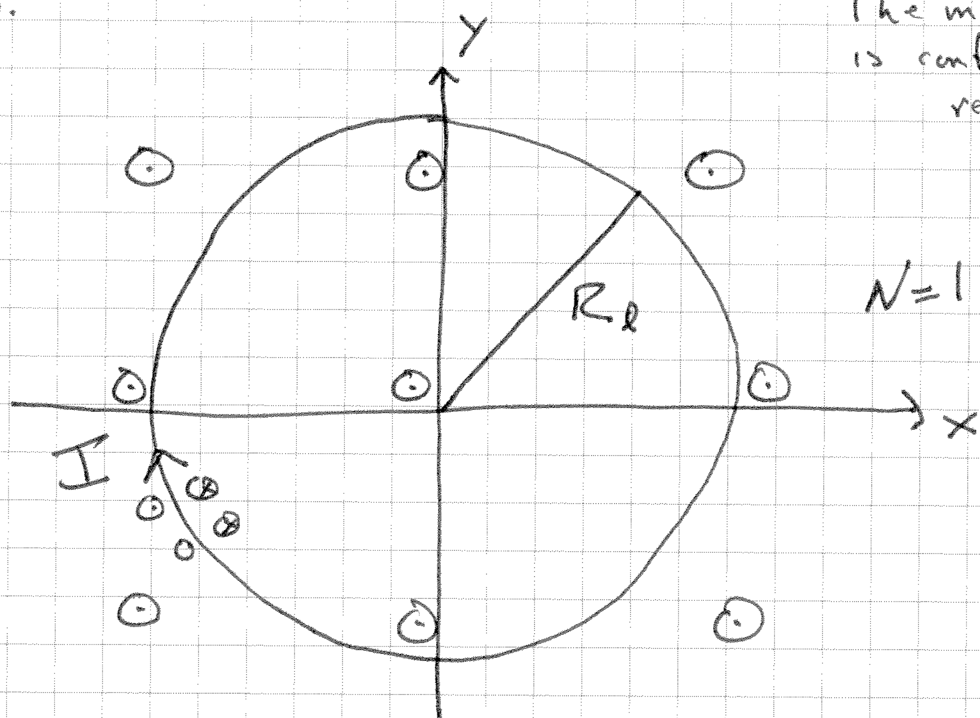
Electromotive Force II

Ex

A loop of radius R_l and cross-sectional area A is in a uniform magnetic field $\vec{B} = \gamma t^2 \hat{z}$. The loop has resistivity ρ and lies in the x-y plane. Compute the field of the induced current at the origin.

The magnetic field is confined to a region $s < b$

$b \gg R$



$N=1$

$\Rightarrow$ The magnetic flux is increasing out of the page. The induced current will produce a flux into the page to oppose the change. By the RHR, the induced current flows clockwise

$\Rightarrow$

Let the curve C that we will apply Faraday's Law to be directed CCW as usual. This produces a positive normal to the surface out of the page in the positive $\hat{z}$ direction by our convention for the normal, $\hat{n} = \hat{z}$

Magnetic Flux

$$\Phi_m = \int_S \vec{B} \cdot d\vec{a} = \vec{B} \cdot \hat{n} \, A_\ell$$

$$\hat{n} = \hat{z}$$

$$A_\ell = \text{Area of Loop} = \pi R_\ell^2$$

$$\Phi_m = \gamma t^2 \pi R_\ell^2$$

Faraday's Law

$$emf = -\frac{d\Phi_m}{dt} = \cancel{\gamma t^2 \pi} \; -2\gamma t \pi R_\ell^2$$

$$= \int_c \vec{E} \cdot d\vec{\ell}$$

$\Rightarrow$ Current will flow in the direction of $\vec{E}$, so the negative sign indicate the current flow is clockwise as we found using Lenz' Law.

The resistance of the loop, R_L, is

$$R_L = \frac{\rho l}{A} = \frac{2\pi R_l \rho}{A}$$

A = cross-sectional area of wire

The current is by Ohm's Law

$$I = \frac{emf}{R_L} = \frac{-2\gamma\pi R_l^2 t}{2\pi R_l \rho / A}$$

$$= \frac{-\gamma A R_l t}{\rho} \quad (- \text{ indicates CW})$$

Current Density in Loop

$$J = \frac{I}{A} = \frac{-\gamma R_l t}{\rho}$$

Induced Field in Loop

$$E = \frac{J}{\sigma} = \rho J = -\gamma R_l t$$

Check against Faraday

$$emf = \oint_C \vec{E} \cdot d\vec{l} = 2\pi R_l E = -2\gamma\pi R_l^2 t$$

$$E = -\gamma R_l t \quad \checkmark$$

Requires cylindrical region

The current I produces a field into the page at the center of the loop.

induced field $\quad \vec{B}_0 = \dfrac{-\mu_0 I}{2R_\ell}\hat{z} = \dfrac{-\mu_0 \gamma A \cancel{R_\ell} t}{2 \cancel{R_\ell} \rho}\hat{z}$

$$= \frac{-\mu_0 \gamma A t}{2\rho}\hat{z}$$

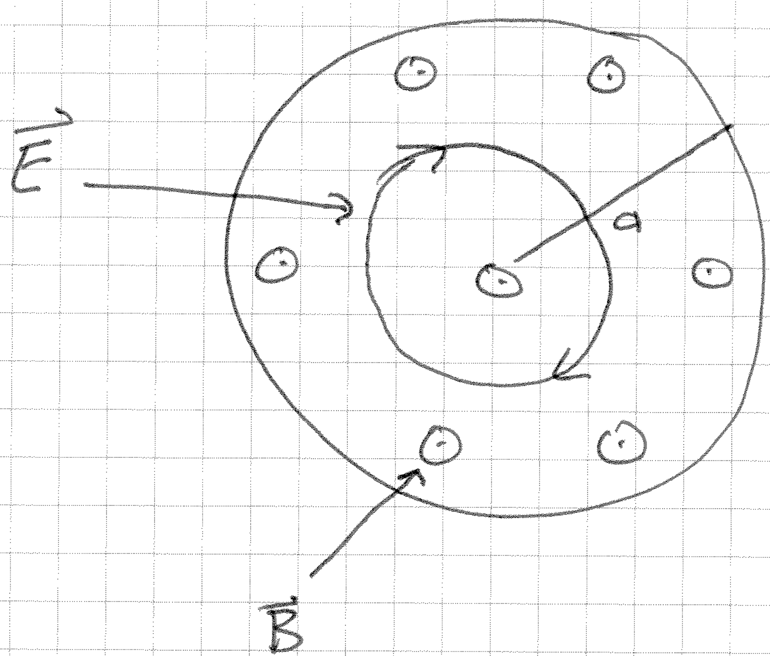
$\Rightarrow$ Suppose we tipped the loop by 30°, what would change?

$$\Phi_m = \vec{B}\cdot\hat{n}\,A_\ell = B A_\ell \cos 30^\circ$$

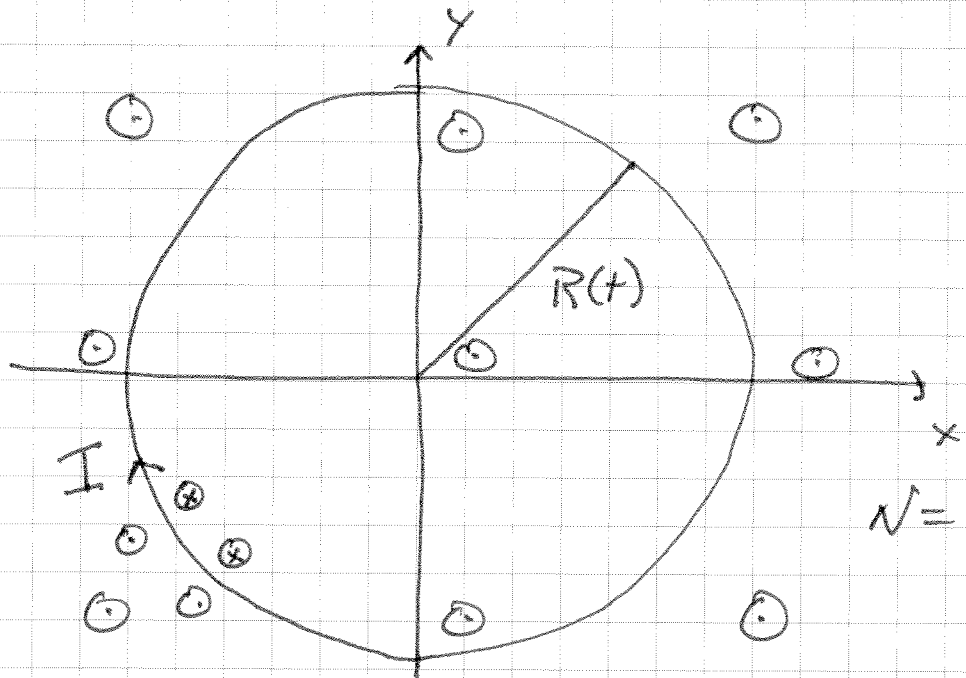
Note, motional emf could not be used to solve the previous problem because there is an induced electric field.

The field we calculated exists whether or not there is a loop of wire. If the field $\vec{B} = \gamma t^2 \hat{z}$ is confined to a circular region $s < a$, the field is circular by symmetry

$$\vec{E} = -\gamma R t \hat{\phi}$$



Ex Place the same loop of wire in a fixed field $\vec{B} = B_0 \hat{z}$, but allow the radius to grow at a constant rate in time, $R(t) = R_0 + vt$.



$\Rightarrow$ Flux out of the page is increasing. The induced current will produce a flux into the page to oppose the change. By the Right Hand Rule, the induced current is clockwise.

Magnetic Flux

$$\Phi_m = N \vec{B} \cdot \hat{n} A_\ell = B_0 \pi R^2(t)$$

Flux Rule (Not Faraday)

$$emf = -\frac{d\Phi_m}{dt} = -2 B_0 \pi v (R_0 + vt)$$

Since no changing magnetic field, this may be done with motional emf.

$$\text{emf} = \frac{W}{q} = \int_c (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$\vec{v} \times \vec{B} = -|\vec{v}||\vec{B}|\,\hat{\phi}$$

$$v = \frac{dR}{dt}$$

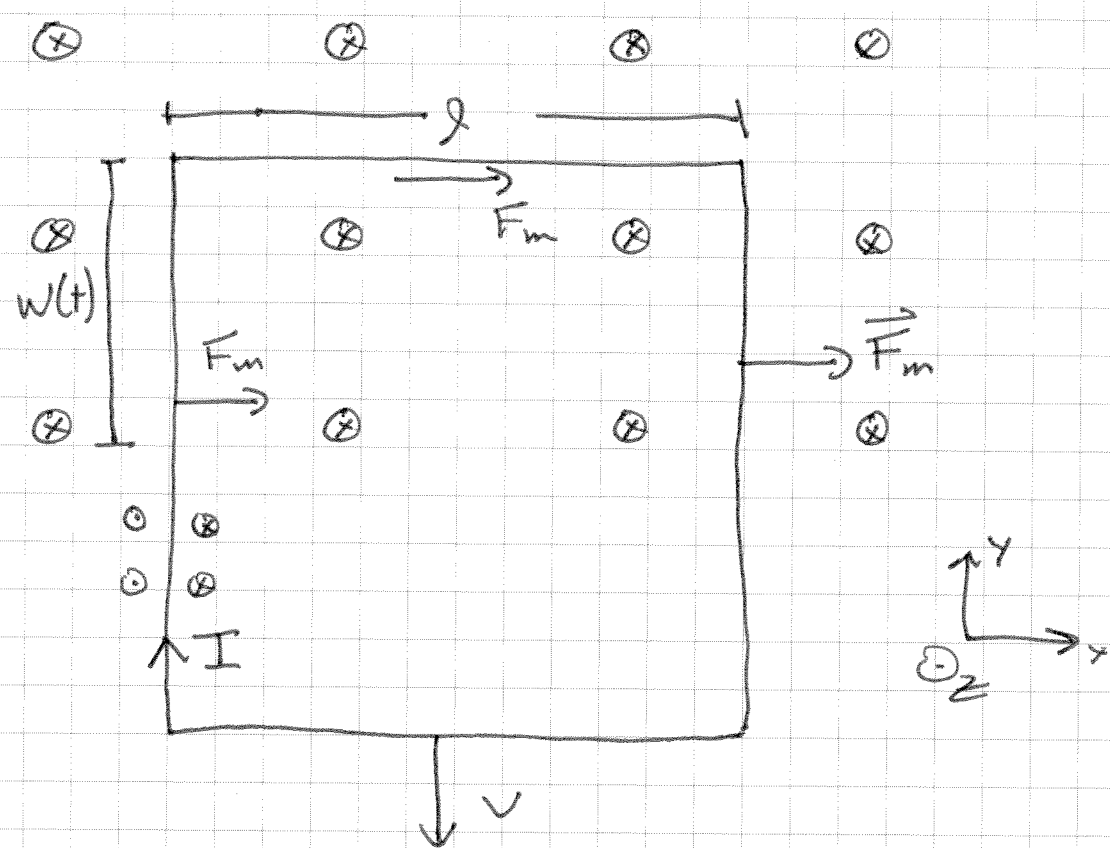
$$\vec{v} \times \vec{B} = -v B_0 \hat{\phi}$$

$$d\vec{l} = R\,d\phi\,\hat{\phi}$$

$$\text{emf} = \oint_c (\vec{v} \times \vec{B}) \cdot d\vec{l} = -v B_0 R \int_0^{2\pi} d\phi$$

$$= -v_0 B_0 (R_0 + vt) 2\pi$$ ✓

Ex Loop pulled out of uniform field $\vec{B} = B_0 \hat{z}$.



$\Rightarrow$ Note, current will flow in the direction of the net magnetic force, $d\vec{F}_m = dq\, \vec{v} \times \vec{B}$

$\Rightarrow$ The flux through the loop is decreasing, so the induced current will produce a flux into the page to oppose the change. The induced current is clockwise by the Right Hand Rule.

Select C to be CCW as usual, $\hat{n} = \hat{z}$.

$$\Phi_m = N \vec{B} \cdot \hat{n} \, A(t) = -N B_0 A(t)$$

$$A(t) = \ell (w_0 - vt) \qquad (v > 0)$$

$$\Phi_m = -N B_0 \ell (w_0 - vt)$$

<u>Flux Rule</u>

$$\text{emf} = -\frac{d\Phi_m}{dt} = -N B_0 \ell v$$

$- \Longrightarrow$ CW opposite C.

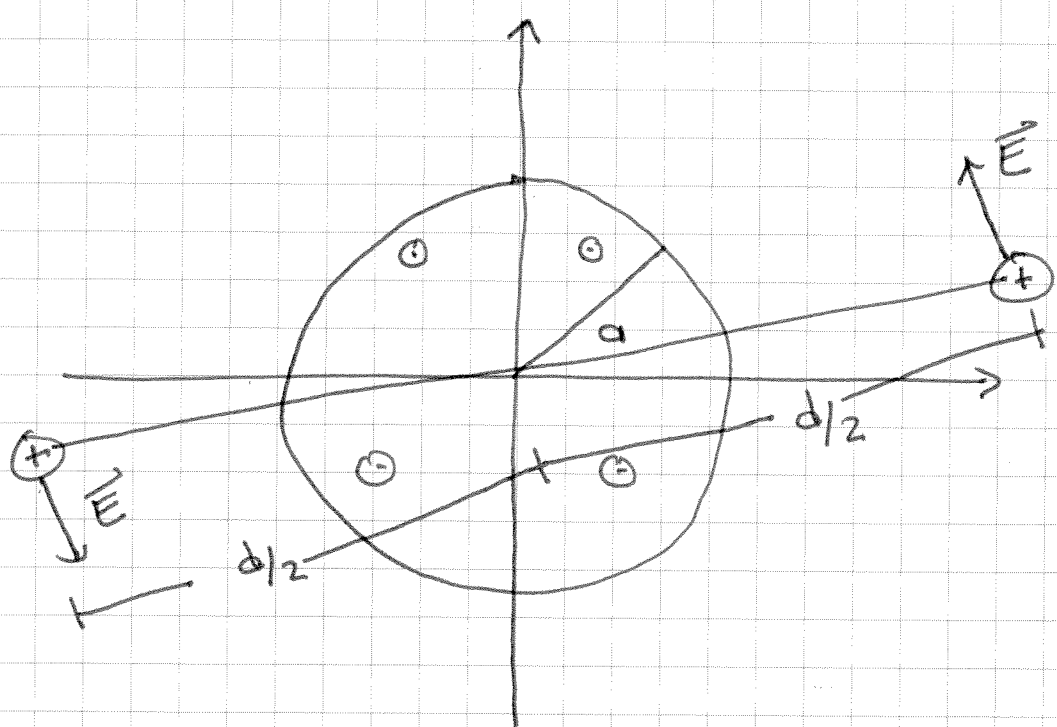
$\Longrightarrow$ Can also calculate from motional emf, only top contributes because only on top is the force along the loop.

$$\text{emf} = N \oint (\vec{v} \times \vec{B}) \cdot d\vec{\ell} = -N v B \ell$$

From $d\ell$ opposite $\vec{F}_m$

Ex Crazy GRE Problem

Let $s < a$ contain a magnetic field $\vec{B} = B_0 \hat{z}$. The field decays from B_0 to 0 in time Δt. Compute the change in angular momentum ΔL of two point charges $+q$ on a stick of length $d > 2a$ free to rotate about the axis of the solenoid.



Torque $\quad T = \dfrac{dL}{dt}$

where L is angular momentum.

$$\Delta L = \int_0^{\Delta t} \tau \, dt$$

By symmetry, the electric field is circular.

The torque is $\tau = 2\tau_q = 2\left(\underbrace{\frac{d}{2}}_{\text{moment arm}} \, q \underbrace{E(d/2)}_{\text{field}}\right)$

$$\tau = d\, q\, E(d/2)$$

<u>Compute $\vec{E}$</u>

$$\text{emf} = \oint \vec{E} \cdot d\vec{l} = 2\pi s\, E(s) = -\frac{d\Phi_m}{dt}$$

$$\Phi_m = NBA = BA = B\,\pi a^2$$

$$\frac{d\Phi_m}{dt} = \pi a^2 \frac{dB}{dt}$$

$$2\pi s\, E(s) = -\pi a^2 \frac{dB}{dt}$$

$$E(s) = \frac{-a^2}{2s} \frac{dB}{dt}$$

At $d/2$

$$E(d/2) = -\frac{a^2}{d}\frac{dB}{dt}$$

therefore the torque is

$$\tau = dq\,E(d/2) = -qa^2\frac{dB}{dt}$$

and the change in angular momentum

$$\Delta L = \int \tau\, dt = -qa^2 \int dB$$

$$= -qa^2 \Delta B = -qa^2 B_0$$