Electromagnetic Waves

Wave equations derived from Maxwell's eqns

$$\nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

We can solve both equations with a wave travelling in the +z direction with

$$\vec{E}(x,t) = \vec{E}_0 e^{i(kz-\omega t)}$$

$$\vec{B}(x,t) = \vec{B}_0 e^{i(kz-\omega t)}$$

where $\vec{E}, \vec{B}$ may be complex and it is understood that we must use the real part for the physical solution. These are called monochromatic (one frequency) plane waves.

If we try these solutions in Maxwell's equations additional constraints emerge.

Gauss' Law

$$\nabla \cdot \vec{E} = 0 \quad \Rightarrow \quad i k E_{z0}\, e^{i(kz - \omega t)} = 0$$

$$\Rightarrow \vec{E}_{0z} = 0$$

No Monopoles

Likewise $\nabla \cdot \vec{B} = 0 \Rightarrow \cancel{B} \; B_{0z} = 0$

$\Rightarrow$ There is no component of the fields in the direction of propagation.

$\Rightarrow \vec{E}_0, \vec{B}_0 \perp$ direction of propagation.

Faraday's Law

$$\nabla \times \vec{E} = \frac{-\partial \vec{E}}{\partial t}$$

$$\cancel{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{0x} & E_{0y} & 0 \end{vmatrix} = \hat{i}\left(-\left(\frac{\partial E_{0y}}{\partial z}\right)\right) - \hat{j}\left(-\right.}$$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{ox} e^{i(kz-\omega t)} & E_{oy} e^{i(kz-\omega t)} & 0 \end{vmatrix}$$

$$= \hat{\imath}\left(-ikE_{oy} e^{i(kz-\omega t)}\right)$$

$$-\hat{\jmath}\left(-ik E_{ox} e^{i(kz-\omega t)}\right)$$

$$= -\frac{\partial \vec{B}}{\partial t} = i\omega\left(\hat{\imath} B_{ox} e^{i(kz-\omega t)} + \hat{\jmath} B_{oy} e^{i(kz-\omega t)}\right)$$

$$-kE_{oy} = \omega B_{ox} \qquad k E_{ox} = \omega B_{oy}$$

Re-write this condition as,

$$\vec{B}_o = \frac{k}{\omega}\left(\hat{z} \times \vec{E}_o\right) = \frac{1}{c}\left(\hat{z} \times \vec{E}_o\right)$$

$$\frac{1}{c}\left(\hat{z}\times\vec{E}_0\right) = \frac{1}{c}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ E_{x0} & E_{y0} & 0 \end{vmatrix}$$

$$= \frac{1}{c}\,\hat{x}\left(-E_{0y}\right) - \frac{1}{c}\,\hat{y}\left(-E_{0x}\right)$$

$$\stackrel{?}{=} B_{0x}\,\hat{x} + B_{0y}\,\hat{y}$$

$$B_{0x} = -\frac{k}{\omega}E_{0y} \qquad B_{0y} = \frac{k}{\omega}E_{0x}$$

✓

By a basic property of the cross-product

$$|\vec{B}_0| = \frac{1}{c}\,|\hat{z}\times\vec{E}_0| = \frac{E_0}{c}$$

since $\vec{E}_0 \perp \hat{z}$.

So the magnitude of $\vec{E}_0$ is $c\vec{B}_0$

We can generalize this a bit. The directions of propagation, the electric field, and the magnetic field are not independent for a wave travelling in a arbitrary direction.

We can write the general form of a wave as

$$\vec{E} = \vec{E}_0 \, e^{i(k_x x + k_y y + k_z z - \omega t)}$$

<u>Dfn - Wave Vector</u>

$$\vec{k} = (k_x, k_y, k_z)$$

$\hat{k}$ is the propagation direction

Consider rotating the coordinates to point in $\hat{k}$ direction, then $\tilde{E} = \hat{E}_0' \, e^{i(kx' - \omega t)}$.

In terms of $\vec{k}$,

$$\vec{E} = \vec{E}_0 \, e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

Dfn Polarization Direction ($\hat{n}$) - The direction of the electric field is the polarization direction

$$\vec{E} = E_0 \hat{n} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

The direction of $\vec{B}$ (generalizing $\hat{z} \times \vec{E}_0$) is

$$\hat{k} \times \hat{n}$$

and $B_0 = E_0 / c$ so

$$\vec{B} = \frac{E_0}{c} e^{i(\vec{k}\cdot\vec{r} - \omega t)} (\hat{k} \times \hat{n})$$

$$= \frac{1}{c} \hat{k} \times \vec{E}$$