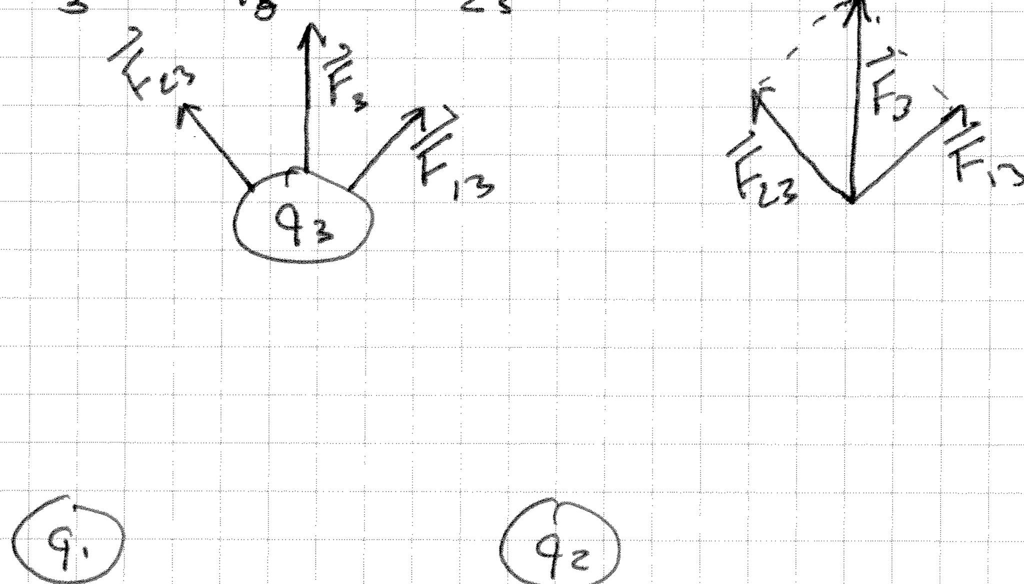
Force and Pressure

Systems with net charge exert forces both on the constituents of the system and on other systems of net charge.

For a system of point charges q_1, q_2, q_3 we can calculate the force exerted on q_3 in a number of ways.

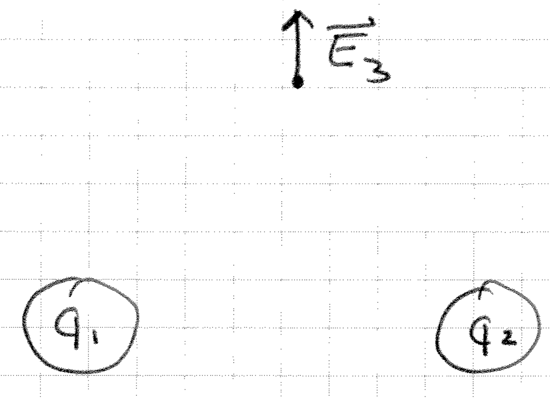
I. Calculate the force q_1 and q_2 exert on q_3 individually and the add

$$\vec{F}_3 = \vec{F}_{13} + \vec{F}_{23}$$



Note, we do not include $\vec{F}_{33}$ because all internal forces come in pairs (Newton III) so the next internal force on an object is zero.

II. Calculate the total field $\vec{E}_3$ at the location of q_3, $\vec{E}_3 = \vec{E}_{13} + \vec{E}_{23}$, then calculate the force as $\vec{F}_3 = q_3 \vec{E}_3$



This calculation is done by pretending q_3 isn't there.

We can also calculate the force one continuous system of charge exerts on another.

Ex Consider two uniform volume charges Q_1 and Q_2 each with charge density ρ spaced a distance R apart.

Method I Brute Force - Integrate the electric force of a point charge $\vec{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}\hat{r}$ over both volumes.

$$\vec{F}_{12} = \int_{V_1} \rho \, d\tau_1 \int_{V_2} \rho \, d\tau_2 \, \frac{1}{4\pi\varepsilon_0 r_{12}^2}\,\hat{r}_{12}$$

$\vec{r}_{12}$ displacement from dq_1 to dq_2.

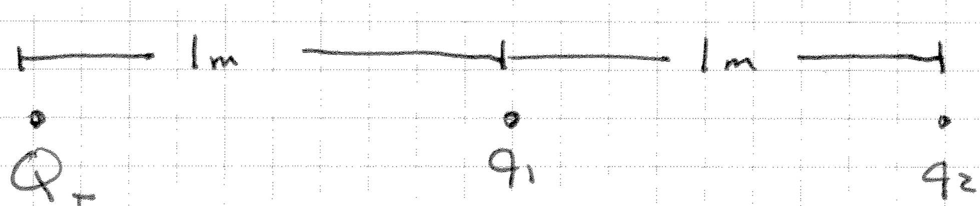
⇒ It always works to chop stuff into small pieces and integrate

⇒ It looks pretty hard.

Is this really necessary? Isn't the force just

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\,\hat{x}$$

⇒ It would be if the field was uniform. Try this reasoning with point charges. Let a point charge Q_T exert a force on two charges $q_1 = q$ and $q_2 = q$ spaced a distance d apart. Compare the actual force F_a to the force you get by concentrating all the charge at the center of charge F_c.



Actual Force

$$\vec{F}_a = \frac{Q_T q_1}{4\pi\varepsilon_0 (1m)^2}\hat{x} + \frac{Q_T q_2}{4\pi\varepsilon_0 (2m)^2}\hat{x}$$

$$= \frac{Q_T q}{4\pi\varepsilon_0} \cdot \left(\frac{5}{4m^2}\right)\hat{x}$$

Force at Center $\quad q_c = 2q$

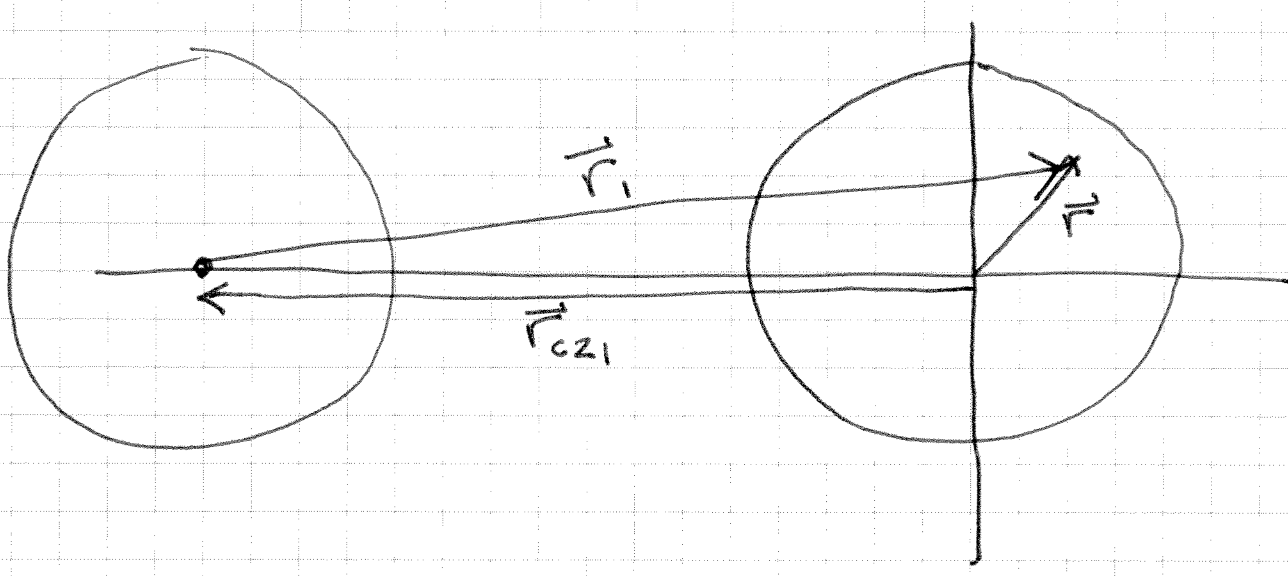
$$\vec{F}_c = \frac{Q_T q_c}{4\pi\varepsilon_0 (3/2\,m)^2}\hat{x} = \frac{Q_T q}{4\pi\varepsilon_0}\left(\frac{8}{9m^2}\right)\hat{x}$$

$$\neq \vec{F}_a$$

$\Rightarrow$ We cannot assume all charge concentrated at center and let force act there.

Back to our two spheres

<u>Method II</u> - The total force due to one part of Q_2 on the other parts must be zero by Newton's III law. So ignore the field of Q_2. Calculate the field on Q_1 at Q_2. Let the center of Q_2 be the origin.



$\vec{r}$ - Vector from center of Q_2 to field point

$\vec{r}_1$ = Vector from center of Q_1 to field point

$\vec{r}_{c21} = -R\hat{x}$ vector from center of Q_2 to center of Q_1

$$\vec{r}_1 = -\vec{r}_{c21} + \vec{r} = R\hat{x} + \vec{r}$$

$$= (x+R,\ y,\ z)$$

The electric field of Q_1 (outside Q_1) is

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0 r_1^2}\hat{r}_1 = \frac{Q_1}{4\pi\varepsilon_0 r_1^3}\vec{r}_1$$

using $\hat{r}_1 = \vec{r}_1 / r_1$

$$Q_1 = \frac{4}{3}\pi a^3 \rho$$

$$r_1 = \sqrt{(x+R)^2 + y^2 + z^2}$$

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0 \left((x+R)^2 + y^2 + z^2\right)^{3/2}}(x+R,\ y,\ z)$$

By symmetry, the ~~field~~ force must point in $+x$ direction, so only look at the x component of field.

$$\vec{F}_{12} = F_x \hat{x} = \int_{V_2} d\tau_2\, \rho\, E_x\, \hat{x}$$

where E_x is the x component of the field.

$$E_x = \frac{Q_1 (x+R)}{4\pi\varepsilon_0 \left((x+R)^2 + y^2 + z^2\right)^{3/2}}$$

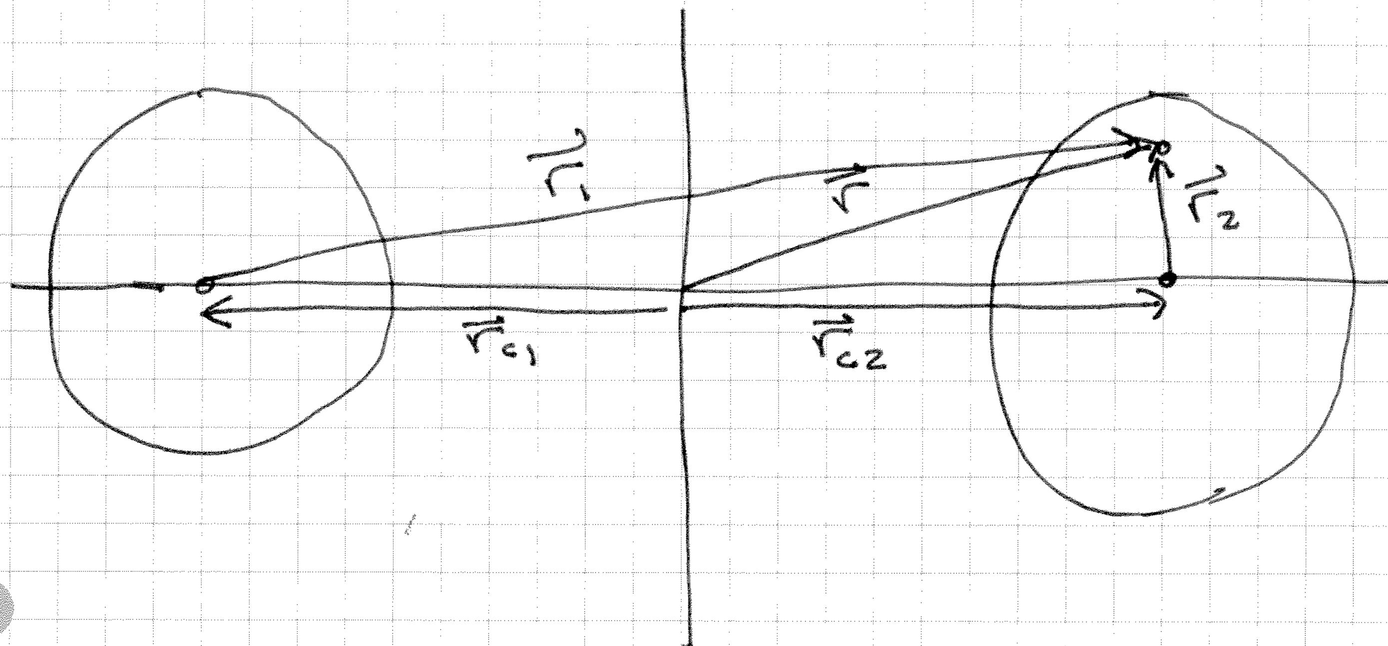
$$F_x = \int_{V_2} d\tau_2 \, \rho \, E_x$$

$$= \frac{Q_1 \rho}{4\pi\varepsilon_0} \int_{V_2} d\tau_2 \, \frac{(x+R)}{\left((x+R)^2 + y^2 + z^2\right)^{3/2}}$$

$$= \frac{Q_1 \rho}{4\pi\varepsilon_0} \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz$$

$$\times \frac{x+R}{\left((x+R)^2 + y^2 + z^2\right)^{3/2}}$$

Method III Calculate the total field of both spheres and use it to calculate the force. We know the self-force of sphere 2 will integrate out.



$$\vec{r}_1 = \vec{r} - \vec{r}_{c1} \qquad \vec{r}_2 = \vec{r} - \vec{r}_{c2}$$

$$\vec{r}_{c1} = -\frac{R}{2}\hat{x} \qquad \vec{r}_{c2} = \frac{R}{2}\hat{x}$$

Inside Q_2

$$\vec{E} = \frac{Q_1(\vec{r}-\vec{r}_{c1})}{4\pi\varepsilon_0|\vec{r}-\vec{r}_{c1}|^3} + \frac{\rho}{3\varepsilon_0}(\vec{r}-\vec{r}_{c2})$$

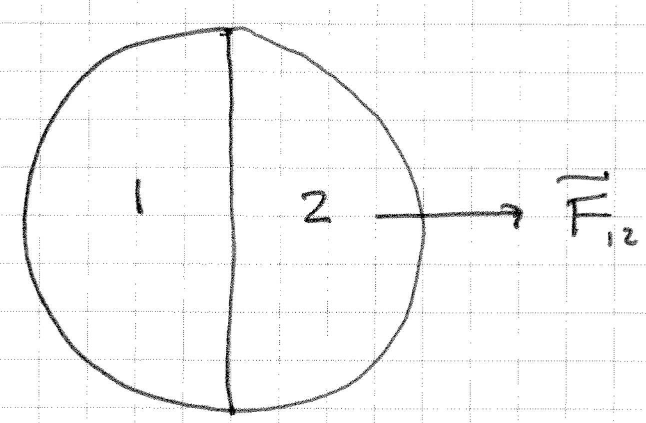
* Field of uniform spherical volume charge is $\rho r\hat{r}/3\varepsilon_0 = \rho\vec{r}/3\varepsilon_0$ by Gauss.

We can then find the force as

$$\vec{F}_{12} = \int_{V_2} d\tau_2 \, \rho \, \vec{E}$$

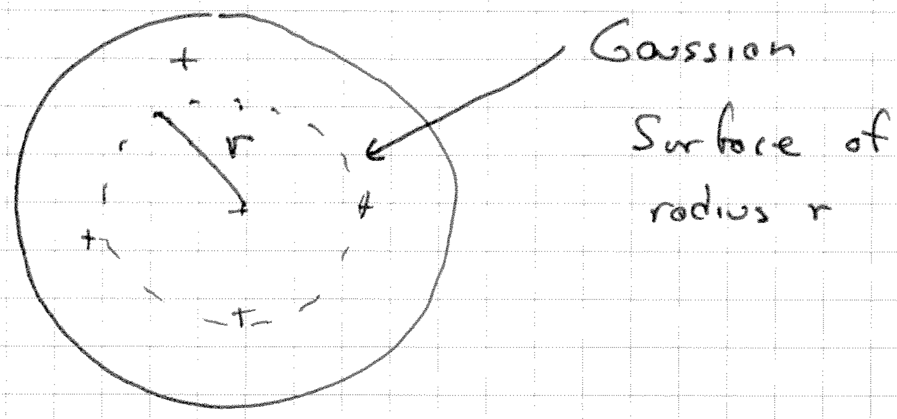
Why would we add this extra term that we know integrates to zero? Sometimes it makes things easier!

<u>Ex.</u> Compute the force one half of a uniformly charged sphere exerts on the other half.



The field of 1, ignoring the field of 2, is complicated, but the field of 1 and 2 combined is simple.

Field of a uniformly charged sphere



$$Q_{enc} = \rho V = \rho \tfrac{4}{3}\pi r^3$$

<u>Gauss' Law</u>

$$\Phi_e = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0} = \frac{\rho}{\varepsilon_0}\frac{4}{3}\pi r^3$$

$$\vec{E} = \frac{r\rho}{3\varepsilon_0}\hat{r} = \frac{\rho\vec{r}}{3\varepsilon_0}$$

The force the left half-sphere exerts on the right half-sphere is then

$$\vec{F} = \int_{V_2} d\tau\, \rho\, E$$

$$\vec{F} = \int_0^{\pi} d\theta \int_{-\pi/2}^{\pi/2} d\phi \int_0^{a} r^2 \sin\theta \, dr \cdot \rho \cdot \left(\frac{\rho \vec{r}}{3\varepsilon_0} \right)$$

Again, only x component survives $\quad \vec{F} = F_x \hat{x}$

$$F_x = \frac{\rho^2}{3\varepsilon_0} \int_0^{\pi} d\theta \int_{-\pi/2}^{\pi/2} d\phi \int_0^{a} r^2 \sin\theta \, dr \, x$$

Put x into spherical coordinates (Griffiths Cover)

$$x = r \sin\theta \cos\phi$$

$$F_x = \frac{\rho^2}{3\varepsilon_0} \int_0^{\pi} d\theta \int_{-\pi/2}^{\pi/2} d\phi \int_0^{a} r^3 \sin^2\theta \cos\phi \, dr$$

integrals separate

$$\int_0^{\pi} d\theta \sin^2\theta = \frac{\pi}{2}$$

$$\int_{-\pi/2}^{\pi/2} \cos\phi \, d\phi = 2$$

$$\int_0^{a} r^3 dr = \frac{a^4}{4}$$

$$F_x = \left(\frac{\rho^2}{3\varepsilon_0} \right) (2) \left(\frac{\pi}{4} \right) \left(\frac{a^4}{4} \right) = \frac{\rho^2 \pi a^4}{24 \varepsilon_0}$$

Check Units

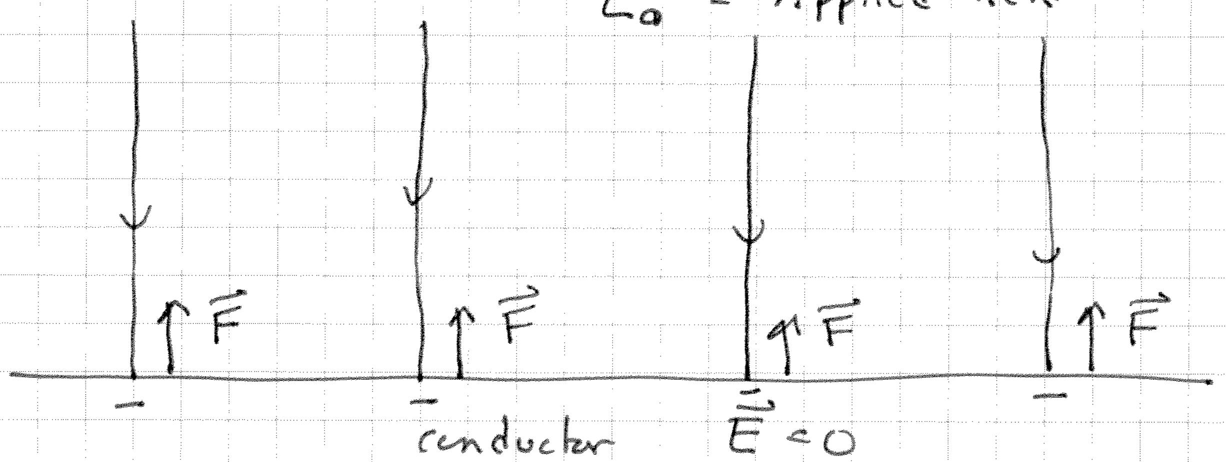
$$\left[\frac{\rho^2 a^4}{\varepsilon_0}\right] = \frac{(C/m^3)^2 (m)^4}{C^2/Nm^2} = N \checkmark$$

The quantity $dq\,\vec{E} = d\tau\,\rho\,\vec{E}$ is a force per unit volume.

The quantity $dq\,\vec{E} = da\,\sigma\,\vec{E}$ is a force per unit area, an electric pressure.

Consider a planar conductor in a uniform electric field.

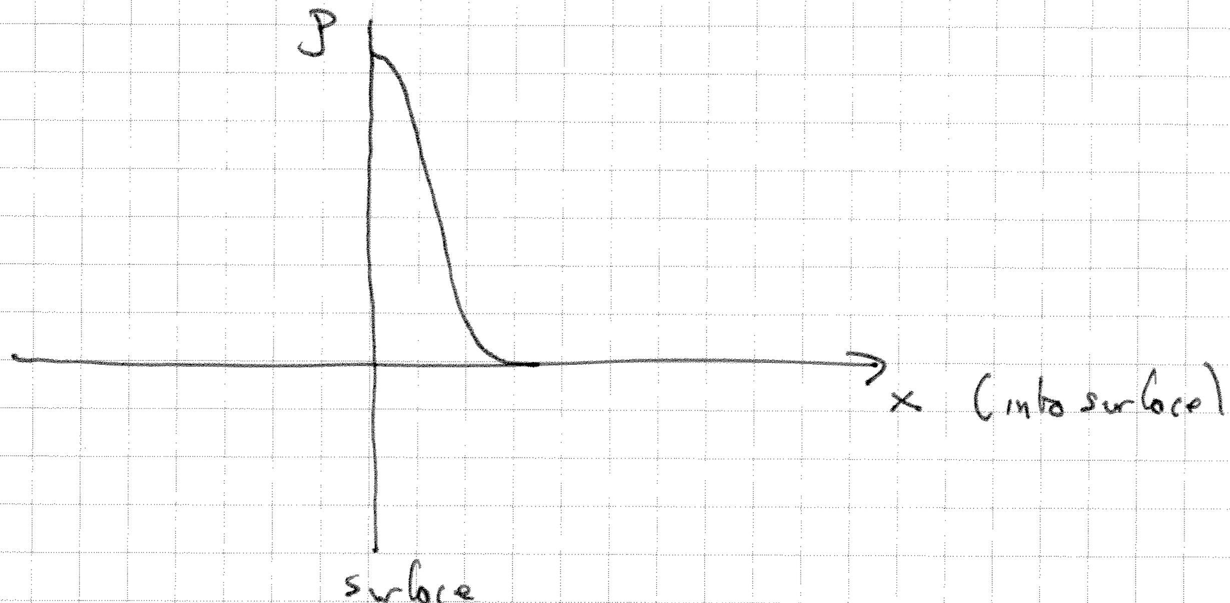
$\vec{E}_a$ = Applied field



$\Rightarrow$ Field exerts force on induced surface charge.

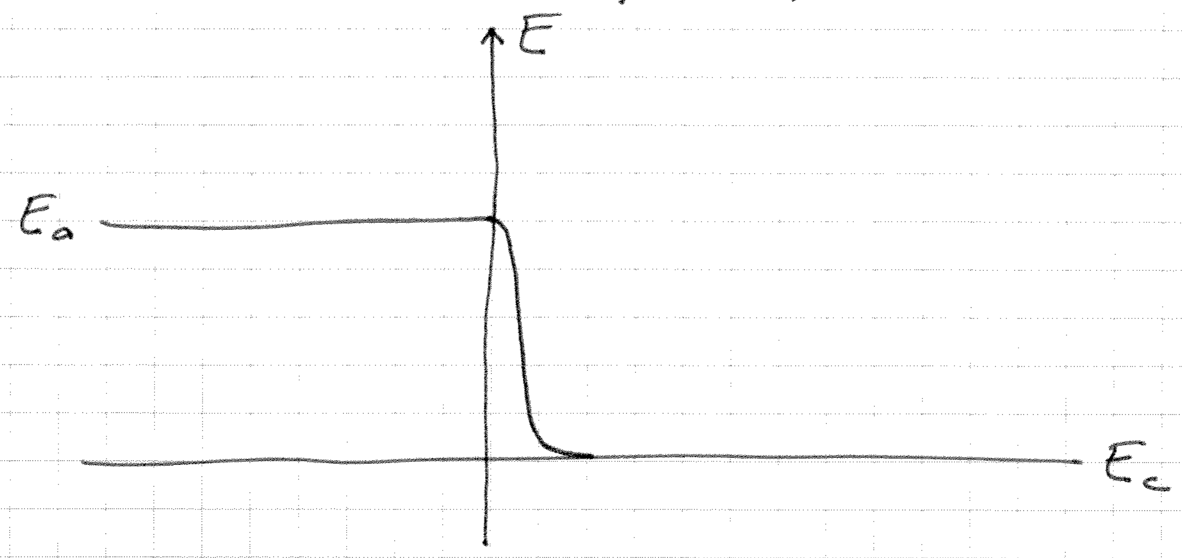
The field exerts a force on the surface, but the field changes value discontinuously at the conductor $\vec{E} = \vec{E}_0$ directly above and $\vec{E} = 0$ below the surface. Which field does the charge feel?

Consider the real situation, where the charge is localized to a thin layer at the surface.



$$\sigma = \int \rho \, dx$$

The field also rapidly decays to zero



because it is screened by the surface charge.

The actual force per unit area is

$$\frac{F}{A} = \int \rho \, dx \, E \approx \left[\int \rho \, dx\right] \frac{E_a + E_c}{2}$$

$\Rightarrow$ The force per unit area exerted on a surface is

$$\text{Pressure} = P = \frac{F}{A} = \sigma E_{ave}$$

where E_{ave} is the average of the fields on the two sides of the surface.

$\Rightarrow$ For our conductor, the electric pressure is

$$P = \frac{\sigma E_a}{2}$$