Homework 10

Due Friday 5/2/2014 by 5pm in my box in physics or in the box outside my office if the physics office is locked. If you want the homework graded before the final exam, get the homework to me by noon May 1st.

Griffiths' 4 Problems (3rd Edition numbers are the same)

7.1(a) and (b)

7.4

7.8

7.12

7.13

7.15

7.16

Problem E.10.1 A wire of length 2ℓ runs along the x-axis and is centered at the origin. The wire is thinner in the middle than at the two ends. The cross-sectional area of the wire is given by $A(x) = A_0(a^2 + x^2)$, where A_0 and a are constants. Make the approximation that the current density depends only on x. Compute the resistance of the wire.

Problem E.10.2 The $x - y$ plane is the boundary between two regions $z > 0$ and $z < 0$ with different electric and magnetic fields. The fields below the plane are

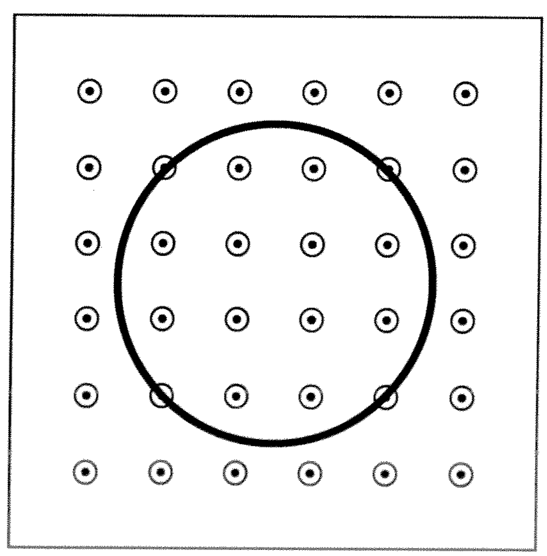
$$\vec{E}_- = \gamma(\hat{x} + 2\hat{y})$$

and

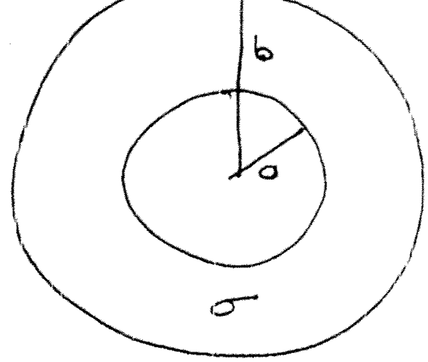
$$\vec{B}_- = \alpha(\hat{x} + \hat{z})$$

The plane has a surface charge density σ and a surface current density $\vec{K} = \Gamma\hat{y}$. Γ, γ, and α are constants. Find the electric and magnetic field above the plane ($z > 0$). Note, while this problem is electro/magnetostatic, it could be altered to include time dependent terms.

Problem E.10.3 A circular ring of conducting wire is in a region with changing magnetic field as shown below. The radius of the ring is increasing as $a(t) = a_1 t^2$, where a_1 is constant. The magnetic field is $B(t) = B_0 t^2$ in the direction drawn with B_0 constant. The conductivity of the wire is σ and it has cross-sectional area A_w. Compute the current flowing the the wire as a function of time and give its direction.



7.1



Assume a current I flows from a → b.

The current density through a surface of radius r

is

$$J = \frac{I}{4\pi r^2}$$

The electric field at a radius r is then given

by Ohm's law

$$E = \frac{J}{\sigma} = \frac{I}{4\pi r^2 \sigma}$$

The potential difference between the shells is

$$\Delta V = -\int_{a\to b} \vec{E}\cdot d\vec{l} = \frac{I}{4\pi\sigma}\int_a^b \frac{dr}{r^2}$$

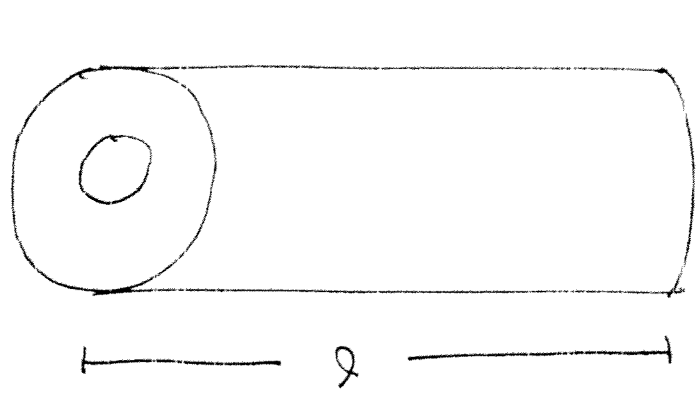
$$|\Delta V| = \frac{I}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{I}{4\pi\sigma ab}(b-a)$$

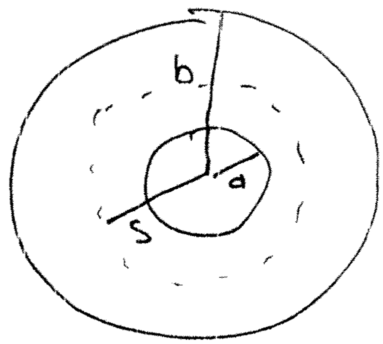
$$I = \frac{4\pi\sigma a b V}{b - a}$$

(b) The resistance is

$$R = \frac{V}{I} = \frac{b-a}{4\pi\sigma a b}$$

7.4





$$\sigma = \frac{K}{s}$$

Let the current flowing between the cylinders be I

The current density flowing through a surface of radius s is then

$$J = \frac{I}{2\pi s \ell}$$

The electric field between the cylinders is

$$E = \frac{J}{\sigma} = \frac{I}{2\pi s \ell \sigma} = \frac{I}{2\pi K \ell}$$

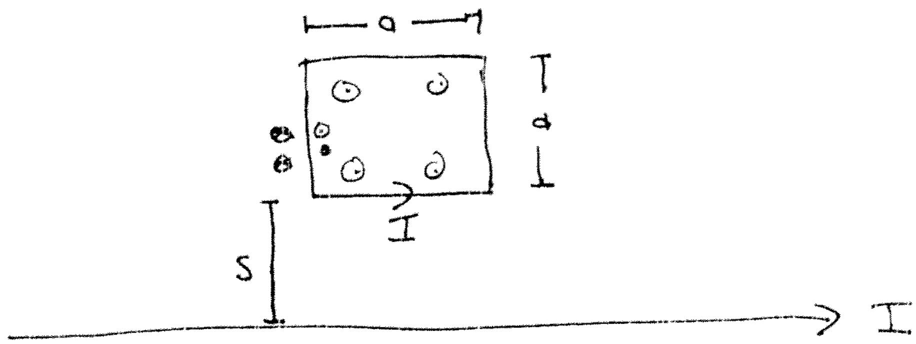
Since the field is constant, the potential difference is

$$\Delta V = E(b-a) = \frac{b-a}{2\pi k \lambda} I$$

So the resistance is

$$R = \frac{\Delta V}{I} = \frac{b-a}{2\pi k \lambda}$$

7.8



(a) Magnetic flux — Slice loop into strips of area $da = a\,ds$

$$\Phi_m = \int B\,da = \int_s^{s+a} \left(\frac{\mu_0 I}{2\pi s} \right) a\,ds$$

where I have used the field of an infinite wire $B = \mu_0 I / 2\pi s$

$$\Phi_m = \frac{\mu_0 I a}{2\pi} \int_s^{s+a} \frac{ds}{s} = \frac{\mu_0 I a}{2\pi} \ln\left(\frac{s+a}{s}\right)$$

(b) By Lenz' law, the magnetic flux out of the page is decreasing as the loop is pulled away so the induced flux is out of the page to oppose the change and the current flows counterclockwise.

If the loop is pulled away at velocity v,

$$s = s_0 + vt$$

$$\Phi_m = \frac{\mu_0 I a}{2\pi} \ln\left(\frac{s_0 + a + vt}{s_0 + vt}\right)$$

$$\text{emf} = \frac{-d\Phi_m}{dt} = -\frac{\mu_0 I a v}{2\pi}\left[\frac{1}{s_0 + a + vt} - \frac{1}{s_0 + vt}\right]$$

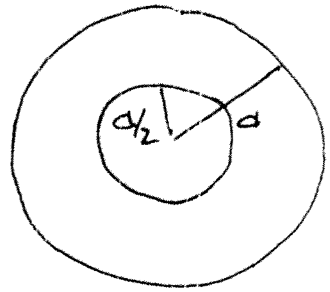
Write this in terms of $s = s_0 + vt$

$$\text{emf} = -\frac{\mu_0 I a v}{2\pi}\left[\frac{1}{s+a} - \frac{1}{s}\right]$$

$$= \frac{\mu_0 I a^2 v}{2\pi s(s+a)}$$

(c) The flux is not changing, so emf = 0.

7.12



The field in the solenoid is given as $B(t) = B_0 \cos\omega t$

The flux through the loop is

$$\Phi_m = BA = \frac{B\pi a^2}{4} = \frac{B_0 \pi a^2}{4}\cos\omega t$$

The emf is by Faraday's law,

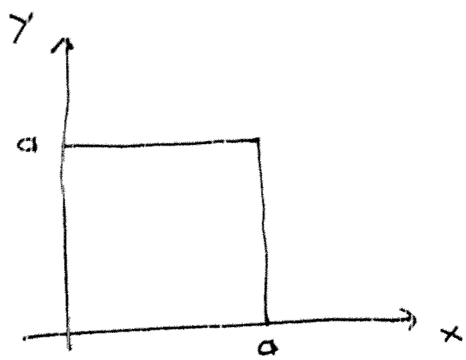
$$emf = -\frac{d\Phi_m}{dt} = \frac{B_0 \pi a^2 \omega}{4}\cos\omega t$$

and the current

$$I = \frac{emf}{R} = \frac{B_0 \pi a^2 \omega}{4R}\cos\omega t$$

7.13

$$B = k y^3 t^2$$



$$da = a\,dy$$

Magnetic Flux

$$\Phi_m = \int B\,da = \int_0^a B\,a\,dy$$

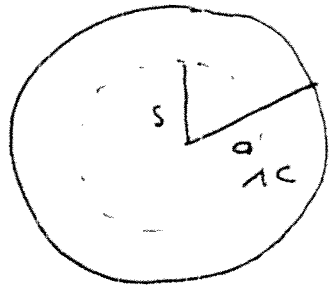
$$= k a t^2 \int_0^a y^3\,dy$$

$$= \frac{k a t^2 a^4}{4} = \frac{k a^5 t^2}{4}$$

emf by Faraday

$$\text{emf} = -\frac{d\Phi_m}{dt} = -\frac{k a^5 t}{2}$$

7.15



n turns per length

Magnetic field in solenoid

$$B(t) = \mu_0 n I(t)$$

Magnetic Flux through surface of radius $s < a$

$$\Phi_m = BA = B\pi s^2 = \mu_0 n \pi s^2 I(t)$$

Faraday's Law

$$\text{emf} = \oint \vec{E} \cdot d\vec{l} = 2\pi s E = -\frac{d\Phi_m}{dt} = -\mu_0 n \pi s^2 \frac{dI}{dt}$$

$$E = -\frac{\mu_0 n s}{2} \frac{dI}{dt} \quad \text{ccw}$$

inside solenoid

$$\vec{E} = \frac{-\mu_0 n s}{2} \frac{dI}{dt} \hat{\phi}$$

Outside Solenoid

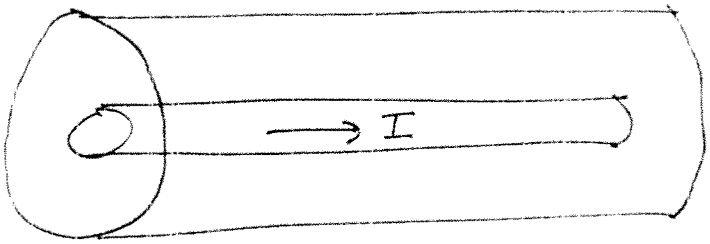
$$\Phi_m = \mu_0 n \pi a^2 I(t)$$

Faraday

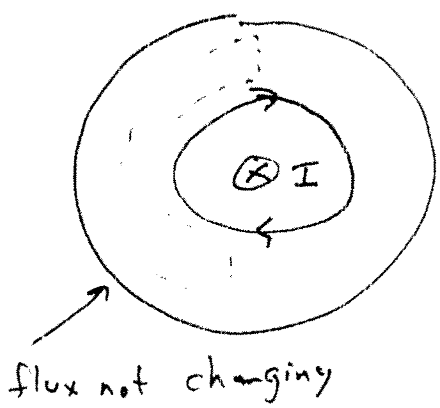
$$2\pi s\, E = -\frac{d\Phi_m}{dt} = -\mu_0 n \pi a^2 \frac{dI}{dt}$$

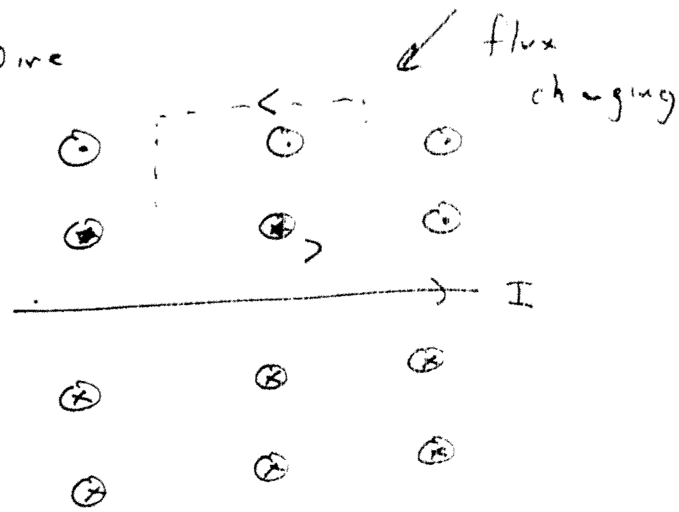
$$\vec{E} = \frac{-\mu_0 n a^2}{2s} \frac{dI}{dt}$$

7.16



Magnetic Field Circles Wire



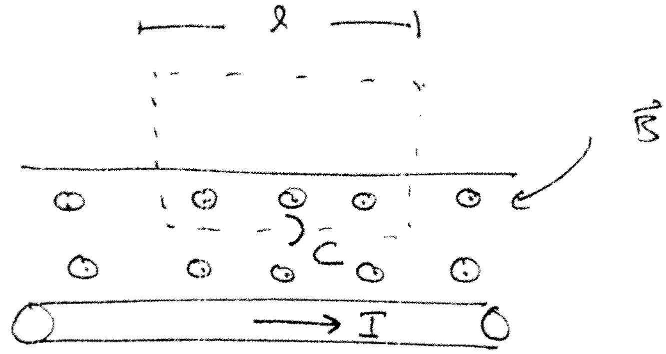


$$\vec{B} = B(t)\,\hat{\phi}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{dB}{dt}\,\hat{\phi} = \left(\frac{\partial E_s}{\partial z} - \frac{\partial E_z}{\partial z}\right)\hat{\phi}$$

$\vec{E}$ cannot be radial $(\vec{E} = E\,\hat{s})$ because no net charge is developed on the wire, therefore the field must point along the axis –

$$\vec{E} = E(t)\,\hat{z}$$



Magnetic Flux through C ~~Field~~

$$\Phi_m = \cancel{BA} \int B\,da \qquad da = l\,ds$$

Magnetic field Ampere's Law

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} = 2\pi s B$$

$$B = \frac{\mu_0 I}{2\pi s}$$

Magnetic Flux

$$\Phi_m = \int B\,da = \int_s^a B\,l\,ds$$

$$= \int_s^a \frac{\mu_0 I}{2\pi s}\,l\,ds = \frac{\mu_0 I l}{2\pi} \ln\left(a/s\right)$$

The field outside is zero.

$$\Phi_m = \frac{\mu_0 l I_0}{2\pi} \ln\left(\frac{a}{s}\right) \cos\omega t$$

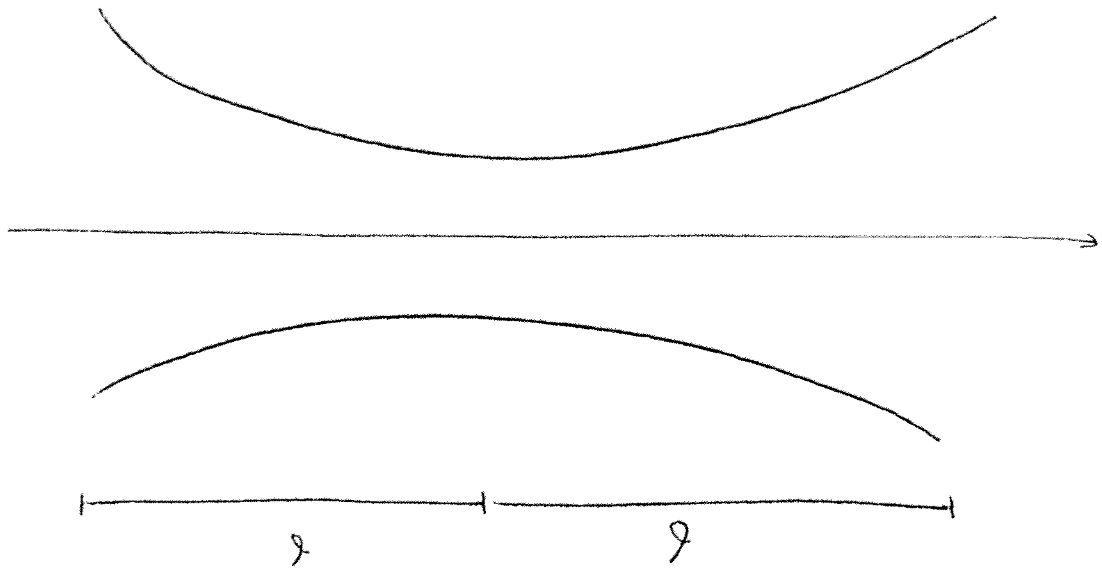
$$\frac{d\Phi_m}{dt} = -\frac{\mu_0 l I_0 \omega}{2\pi} \sin\omega t \ln\left(a/s\right)$$

<u>Faraday's Law</u>

$$\oint \vec{E}\cdot d\vec{l} = -\overset{=0}{E_{outside}}\, l + E_{inside}\, l = -\frac{d\Phi_m}{dt}$$

$$\vec{E}_{inside} = \frac{\mu_0 I_0 \omega}{2\pi} \sin\omega t \ln\left(a/s\right) \hat{z}$$

E.10.1



Assume a current I flows through the wire.

The current density is

$$J = \frac{I}{A}$$

This is related to the field by

$$J = \sigma E$$

$$E = \frac{\sigma I}{A}$$

~~The potential difference along the wires is~~

$$\cancel{\Delta V = -\int_{-l}^{l} E\,dx = \sigma I \int_{-l}^{l} \frac{dx}{A_0 + x^2}}$$

The potential difference across the ends of the wire is given by

$$\Delta V = -\int_{-l}^{l} \vec{E} \cdot d\vec{l}$$

$$= -\sigma I \int_{-l}^{l} \frac{dx}{A(x)}$$

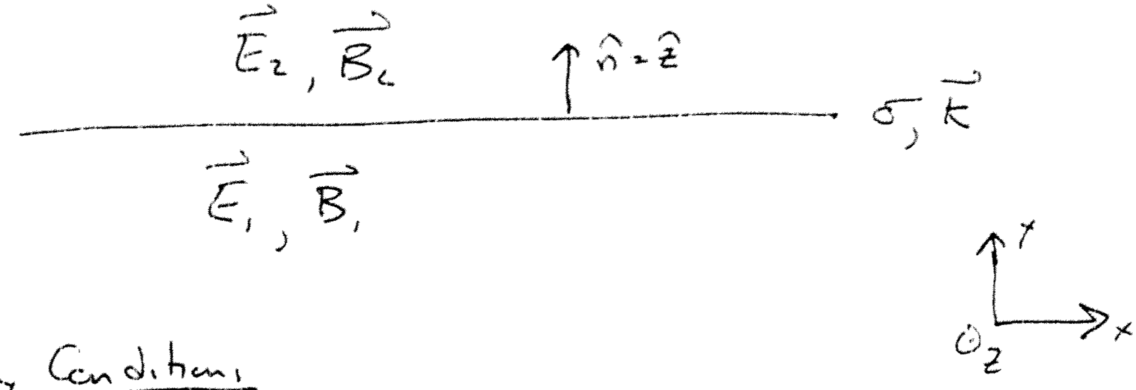
$$= \frac{-\sigma I}{A_0} \int_{-l}^{l} \frac{dx}{a^2+x^2}$$

$$= \frac{-2\sigma I}{A_0} \int_{0}^{l} \frac{dx}{a^2+x^2} = \frac{-2\sigma I}{a A_0} \tan^{-1}\left(\frac{l}{a}\right)$$

The resistance is given by Ohm's law

$$R = \frac{\Delta V}{I} = \frac{2\sigma}{a A_0} \tan^{-1}\left(l/a\right)$$

~~2~~ E.10.2



Boundary Conditions

$$E_2^{\perp} - E_1^{\perp} = \sigma/\epsilon_0$$

$$E_{2z} = E_{1z} + \sigma/\epsilon_0 = \sigma/\epsilon_0$$

$$\vec{E}_2^{\parallel} = \vec{E}_1^{\parallel} \qquad \vec{E}_2^{\parallel} = \gamma\hat{x} + 2\gamma\hat{y}$$

$$\boxed{\vec{E}_2 = \gamma\hat{x} + 2\gamma\hat{y} + \frac{\sigma}{\epsilon_0}\hat{z}}$$

$$B_2^{\perp} = B_1^{\perp} \Rightarrow B_2^{\perp} = \alpha\hat{z}$$

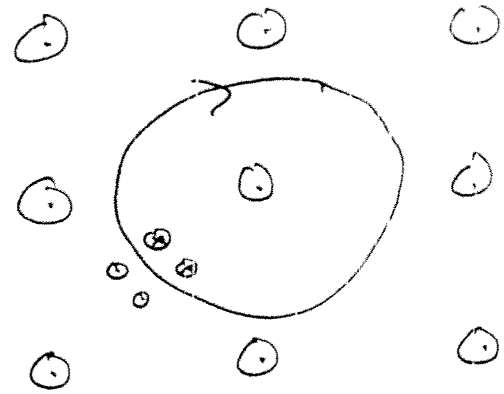
$$\vec{B}_2^{\parallel} - \vec{B}_1^{\parallel} = \mu_0(\vec{K}\times\hat{n}) = \mu_0(\Gamma\hat{y}\times\hat{z}) = \mu_0\Gamma\hat{x}$$

$$\vec{B}_2^{\parallel} = \vec{B}_1^{\parallel} + \mu_0\Gamma\hat{x} = \alpha\hat{x} + \mu_0\Gamma\hat{x} = (\alpha+\mu_0\Gamma)\hat{x}$$

$$\vec{B}_2 = (\alpha+\mu_0\Gamma)\hat{x} + \alpha\hat{z}$$

E.10.3

~~4~~5 To oppose an increasing flux out of the page, the induced current must flow clockwise



The resistance of the loop is $R = \dfrac{2\pi a}{\sigma A_w}$

The flux through the loop is

$$\Phi_m = BA = (B_o t^2)\pi(a_1 t^2)^2$$

$$= B_o a_1^2 \pi t^6$$

The emf around the loop is

$$emf = -\frac{d\Phi_m}{dt} = -6 B_o a_1^2 \pi t^5$$

The current is then

$$I = \frac{emf}{R} = \frac{6 B_o a_1^2 \pi t^5}{2\pi a_1 t^2/\sigma A_w} = 3\sigma A_w a_1 t^3 B_o$$