Homework 3

Due Friday 2/7/2014 - at beginning of class

Reading Assignment - Chapter 2.4-2.5

Griffiths Problems, 4th Edition

Each problem should be started on its own piece of paper. Points will be removed from solutions that are difficult to read.

When the problem numbering is different between the 3rd and 4th edition of Griffiths, the third edition number is in parenthesis.

2.23

2.41 (Griffiths 3rd Edition problem 2.37)

2.42 (Griffiths 3rd Edition problem 2.38)

2.43 (Griffiths 3rd Edition problem 2.39)

Problem 1.1 A spherical system has a NON-UNIFORM volume charge density $\rho = \gamma/r$ for $r < a$ and $\rho = 0$ for $r > 0a$ where γ is a constant. Compute either the electric potential everywhere or the total energy of the system.

Problem 1.2 Calculate the electric potential at a point $(0, 0, z)$ of a finite cylinder ($s < a$, $0 < z < \ell$) containing a uniform volume charge density ρ. Work on the problem until you have a single one-dimensional integral to do.

Problem 1.3 An infinite cylinder of radius a contains a uniform volume charge density ρ. Compute the potential difference between a point on the axis and a point on the outside surface.

Problem 1.4 Calculate the field and potential at the origin of a NON-UNIFORM spherical volume charge of radius a and charge density $\rho = \gamma \sin(2\theta)$ where γ is constant.

Problem E.3.5 Two spherical shells of radius a and b, $a < b$, have uniformly distributed charges $Q_a = Q$ and $Q_b = -Q$. Compute the energy between the shells.

Problem 1.6 The electric potential in some region of space is $V = V_0 x^2 - V_0 y^2$. Compute the electric field in cylindrical coordinates. What is the charge density in the region containing the field?

(2.28)

$$\vec{E} = \begin{cases} 0 & r < a \\ \dfrac{k(r-a)}{r^2 \varepsilon_0} \hat{r} & a < r < b \\ \dfrac{Q_T}{4\pi\varepsilon_0 r^2} \hat{r} & r > b \end{cases}$$

$$V = -\int E \, dr$$

$$V = \frac{Q_T}{4\pi\varepsilon_0 r} + C_{III} \quad (r > b)$$, point charge field.

$$V = C_I \qquad r < a$$, since $E = 0$

$$V = -\frac{k}{\varepsilon_0} \int \left(\frac{1}{r} - \frac{a}{r^2} \right) dr \qquad a < r < b$$

$$= -\frac{k}{\varepsilon_0} \left(\ln(r) + \frac{a}{r} \right) + C_{II}$$

where C_I, C_{III}, C_{II} are constants of integration.

The potential must be continuous.

Select C_{III} so $V(\infty)=0 \Rightarrow C_{III}=0$

$$V(b) = -\frac{k}{\varepsilon_0}\left(\ln(b)+\frac{a}{b}\right)+C_{II} = \frac{Q_T}{4\pi\varepsilon_0 b}$$

$$C_{II} = \frac{Q_T}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\ln(b) + \frac{k}{\varepsilon_0}\frac{a}{b}$$

$$V(r) = \frac{Q_T}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\left(\ln(b)-\ln(r)\right) + \frac{ka}{\varepsilon_0}\left(\frac{1}{b}-\frac{1}{r}\right)$$

$$= \frac{Q_T}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\ln\left(\frac{b}{r}\right) + \frac{ka}{\varepsilon_0 br}(r-b)$$

$a<r<b$

Match potential at $r=a$

$$V(a) = C_I = \frac{Q_T}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\ln\left(\frac{b}{a}\right) + \frac{ka}{\varepsilon_0 ba}(a-b)$$

$$Q_T = 4\pi k (b-a)$$

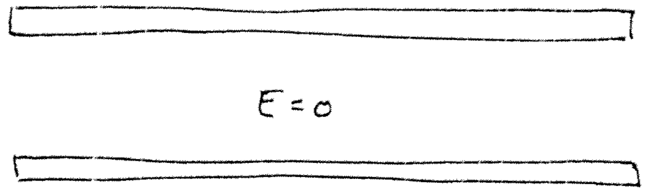
$$V(a) = \frac{4\pi k (b-a)}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\ln\left(\frac{b}{a}\right) + \frac{k}{\varepsilon_0 b}(a-b)$$

$$= \frac{k}{\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

2.41

~~2.37~~

$$E = \frac{\sigma}{\varepsilon_0}$$



$$\sigma = \frac{Q}{A}$$

$$E = 0$$

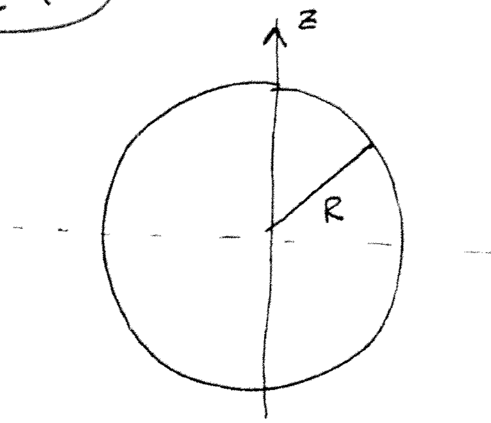
$$\sigma = Q/A$$

$$E = \frac{\sigma}{\varepsilon_0}$$

$$P = \sigma \frac{E}{2} = \frac{\sigma \cdot \sigma/\varepsilon_0}{2}$$

$$= \frac{\sigma^2}{2\varepsilon_0} = \frac{Q^2}{2A^2\varepsilon_0}$$

~~2.38~~ 2.42



Electric field $\qquad \vec{E} = 0 \qquad r < R$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad r > R$$

Electric Pressure

$$\vec{P} = \sigma \vec{E}_{ave} = \frac{\sigma}{2}\vec{E} \quad r > R$$

$$= \frac{\sigma Q}{8\pi\varepsilon_0 R^2}\hat{r} \qquad \sigma = \frac{Q}{4\pi R^2}$$

Total force exerted on northern hemisphere

$$\vec{F} = \int \vec{P}\, da$$

$$da = R\,d\theta \sin\theta\, R\,d\phi = R^2 \sin\theta\, d\theta\, d\phi$$

Evidently only $\hat{z}$ component survives,

$$\hat{r} = \underbrace{\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y}}_{\text{integrates to zero}} + \cos\theta\,\hat{z}$$

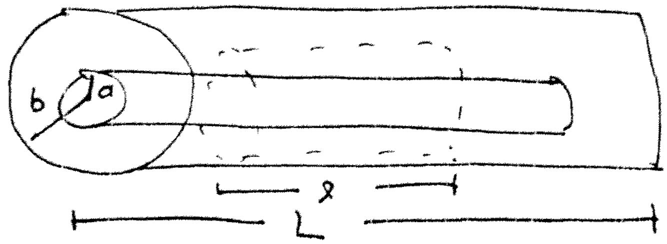
$$\vec{F} = \int \vec{P}\,da = \int \left(\frac{\sigma Q}{4\pi\varepsilon_0 R^2} \cdot \cos\theta\,\hat{z} \right) \left(R^2 \sin\theta\, d\theta\, d\phi \right)$$

$$= \frac{\sigma Q}{8\pi\varepsilon_0} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi/2} d\theta \cos\theta \sin\theta}_{\frac{1}{2}}\ \hat{z}$$

$$= \frac{\sigma Q\,\pi}{8\pi\varepsilon_0}\hat{z} = \frac{\sigma Q}{8\varepsilon_0}\hat{z}$$

$$= \frac{Q}{8\varepsilon_0}\left(\frac{Q}{4\pi R^2}\right) = \frac{Q^2}{32\pi\varepsilon_0 R^2}\,\hat{z}$$

~~2.37~~ 2.43



Add a charge Q to inner conductor, and charge $-Q$ to outer conductor.

Use a cylindrical Gaussian surface that encloses the inner conductor.

$$Q_{enc} = \lambda l$$

where $\lambda = Q/L$

Gauss Law

$$\Phi = 2\pi r l E = \frac{Q_{enc}}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi r \varepsilon_0}$$

Potential Difference

$$V = -\int_a^b E\,dr$$

$$= \frac{-\lambda}{2\pi\varepsilon_0}\ln(b/a) = -\frac{Q}{2\pi\varepsilon_0 L}\ln(b/a)$$

Capacitance

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{\frac{Q}{2\pi\varepsilon_0 L}\ln(b/a)}$$

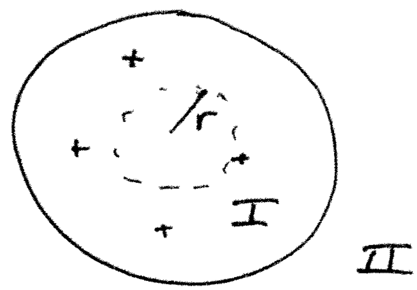
$$= \left(\frac{2\pi\varepsilon_0}{\ln(b/a)}\right)L$$

Capacitance per unit length

E 8.1

A sphere has non-uniform volume charge density $\rho = \gamma / r$ for $r < a$ and 0 for $r > a$.

Compute the total energy of the sphere.



Spherical Gaussian Surface Radius r.

<u>Region I</u>

$$Q_{enc} = \int_0^r 4\pi r^2 \rho \, dr = 4\pi\gamma \int_0^r dr \, r$$

$$= 4\pi \gamma \frac{r^2}{2} = 2\pi\gamma r^2$$

$$\vec{E}_I = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{2\pi\gamma r^2}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{\gamma}{2\varepsilon_0}\hat{r}$$

<u>Region II</u> $\quad Q_{enc} = 2\pi\gamma a^2$

$$\vec{E}_{II} = \frac{2\pi\gamma a^2}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{\gamma a^2}{2\varepsilon_0 r^2}\hat{r}$$

Potential

$$V_I = -\int E_I \, dr = \frac{-\gamma r}{2\varepsilon_0} + C_I$$

$$V_{II} = -\int E_{II} \, dr = \frac{\gamma a^2}{2\varepsilon_0 r} + C_{II}$$

$$V_{II}(\infty) = 0 \implies C_{II} = 0$$

Continuity

$$V_I(a) = V_{II}(a)$$

$$\frac{-\gamma a}{2\varepsilon_0} + C_I = \frac{\gamma a^2}{2\varepsilon_0 a} = \frac{\gamma a}{2\varepsilon_0}$$

$$C_I = \frac{\gamma a}{\varepsilon_0}$$

$$V_I = \frac{-\gamma r}{2\varepsilon_0} + \frac{\gamma a}{\varepsilon_0}$$

$$V_{II} = \frac{\gamma a^2}{2\varepsilon_0 r}$$

Total Energy

$$U = \int_{space} \frac{1}{2}\varepsilon_0 E^2 \, d\tau$$

$$= \int_0^a \frac{1}{2}\varepsilon_0 E^2 d\tau + \int_a^\infty \frac{1}{2}\varepsilon_0 E^2 d\tau$$

Inside sphere (U_I)

$$u_I = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 \left(\frac{\gamma}{2\varepsilon_0}\right)^2$$

$$= \frac{1}{8}\frac{\gamma^2}{\varepsilon_0}$$

Since the energy density is constant, the total energy is the energy density multiplied by the volume.

$$U_I = u_I V = \frac{1}{8}\frac{\gamma^2}{\varepsilon_0} \cdot \frac{4}{3}\pi a^3$$

$$= \frac{1}{6}\frac{\pi \gamma^2 a^3}{\varepsilon_0}$$

The energy density outside is

$$u_{II} = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0\left(\frac{Q^2}{4\pi\varepsilon_0 r^2}\right)^2$$

$$= \frac{1}{32}\frac{Q^2}{\pi^2\varepsilon_0 r^4}$$

where $Q = 2\pi\gamma a^2$ the total charge of the sphre.

angular variables integrated out.

$$U_{II} = \int_a^\infty 4\pi r^2 u_{II}\, dr$$

$$= \int_a^\infty 4\pi \frac{Q^2}{32\pi^2\varepsilon_0 r^2}\, dr$$

$$= \frac{1}{8}\frac{Q^2}{\pi\varepsilon_0}\left(-\frac{1}{r}\right)_a^\infty$$

$$= \frac{1}{8}\frac{Q^2}{\pi\varepsilon_0 a} = \frac{1}{8}\frac{(2\pi\gamma a^2)^2}{\pi\varepsilon_0 a}$$

$$= \frac{1}{2}\frac{\pi\gamma^2 a^3}{\varepsilon_0}$$

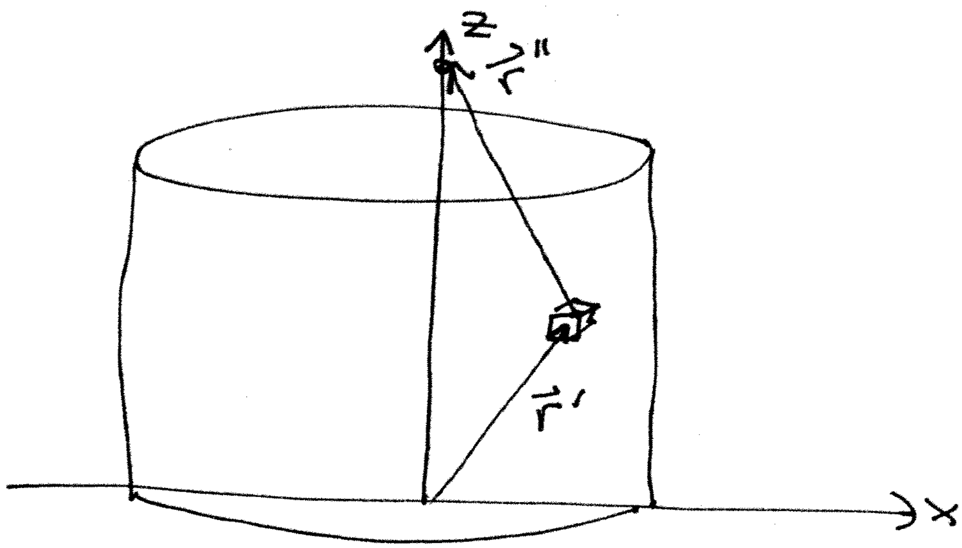
Total Energy

$$U = U_I + U_{II} = \frac{1}{2}\pi\frac{\gamma^2 a^3}{\varepsilon_0} + \frac{1}{6}\pi\frac{\gamma^2 a^3}{\varepsilon_0}$$

$$= \frac{2}{3}\,\frac{\pi\gamma^2 a^3}{\varepsilon_0}$$

E 3.2

Calculate the electric potential of a finite cylinder ($s<a$, $0<z<\ell$) containing a uniform charge density ρ at a point $z>\ell$ along the z-axis.



Field Point $\quad \vec{r} = (0, 0, z)$

Source Point $\quad \vec{r}' = s'\hat{s}' + z'\hat{z}$

Displacement $\quad \vec{r}'' = \vec{r} - \vec{r}'$

$$= -s'\hat{s}' + (z - z')\hat{z}$$

Length $\quad r'' = \sqrt{s'^2 + (z - z')^2}$

Electric Potential

$$V(z) = \int_{\text{cylinder}} \frac{\rho \, d\tau'}{4\pi\epsilon_0 r''}$$

$$d\tau' = (ds')(s' d\phi') \, dz'$$

$$V(z) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^{l} dz' \int_0^{a} ds' \quad \frac{\rho s'}{\sqrt{s'^2 + (z-z')^2}}$$

Integrate $d\phi'$ to give 2π

$$V(z) = \frac{\rho}{2\epsilon_0} \int_0^{l} dz' \int_0^{a} ds' \; \frac{s'}{\sqrt{s'^2 + (z-z')^2}}$$

Integrate s'

$$\int_0^a ds' \; \frac{s'}{\sqrt{s'^2 + (z-z')^2}} = \sqrt{s'^2 + (z-z')^2} \Big|_0^a$$

$$= \sqrt{a^2 + (z-z')^2} \qquad \text{(Schaums)}$$

$$- |z - z'|$$

$$z - z' > 0$$

$$= \sqrt{a^2 + (z-z')^2} - (z - z')$$

$$V(z) = \frac{\rho}{2\varepsilon_0}\int_0^{\ell} dz' \left(\sqrt{a^2+(z-z')^2} - (z-z')\right) .$$

That's as far as the problem asked you to take it, but let's complete it.

$$\int_0^{\ell} dz'(z-z') = z\ell - \ell^2/2$$

$$\int_0^{\ell} \sqrt{a^2+(z-z')^2}\, dz' = \qquad\qquad u = z-z'$$

$$du = -dz'$$

$$= -\int_0^{z-\ell} \sqrt{a^2+u^2}\, du$$

$$\int_z^{z-\ell} du \sqrt{a^2+u^2} = \left(\frac{1}{2}u\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left(u+\sqrt{u^2+a^2}\right)\right)\Bigg|_z^{z-\ell}$$

$$= \frac{1}{2}(z-\ell)\sqrt{(z-\ell)^2+a^2} - \frac{1}{2}z\sqrt{z^2+a^2}$$

$$+ \frac{a^2}{2}\ln\left(z-\ell+\sqrt{(z-\ell)^2+a^2}\right) - \frac{a^2}{2}\ln\left(z+\sqrt{z^2+a^2}\right)$$

$$= \frac{1}{2}(z-\ell)\sqrt{(z-\ell)^2+a^2} - \frac{1}{2}z\sqrt{z^2+a^2}$$

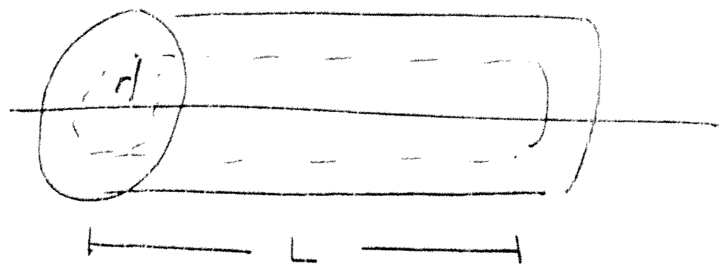
$$+ \frac{a^2}{2}\ln\left(\frac{z-\ell+\sqrt{(z-\ell)^2+a^2}}{z+\sqrt{z^2+a^2}}\right)$$

So the total potential is

$$V(z) = \frac{\rho}{2\varepsilon_0}\left[\frac{1}{2}(z-l)\sqrt{(z-l)^2+a^2} - \frac{1}{2} z\sqrt{z^2+a^2} + \frac{a^2}{2}\ln\left(\frac{z-l+\sqrt{(z-l)^2+a^2}}{z+\sqrt{z^2+a^2}}\right) + \frac{l^2}{2} - zl \right]$$

At least the units are correct; I don't feel like checking the $z \to \infty$ limit.

E3.3



Use a cylindrical Gaussian surface of radius r and length L.

The charge inside the surface is $Q_{enc} = \rho V$

$$= \pi r^2 L \rho$$

The flux out of the cylinder is

$$\Phi = E A_s = 2\pi r L E = \frac{Q_{enc}}{\varepsilon_0} \quad +5$$

$$E = \frac{Q_{enc}}{2\pi r L \varepsilon_0} = \frac{\pi r^2 L \rho}{2\pi r L \varepsilon_0} \overset{+5}{=} \frac{r\rho}{2\varepsilon_0} \quad +5$$

The potential difference between $r=0$ and $r=a$ is

$$\Delta V_{0a} = -\int_0^a E\,dr \overset{+5}{=} -\frac{\rho r^2}{4\varepsilon_0}\Big|_0^a = -\frac{\rho a^2}{4\varepsilon_0} \quad +5$$

where the sign is correct because the potential decreases in the direction of the field.

E 8.4

Calculate the field and potential at the origin of a NON-UNIFORM spherical volume charge of radius a and charge density

$$\rho = \gamma \sin 2\theta$$

where γ is a constant

Compute Field

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_{\text{volume}} \frac{\rho \, d\tau}{r''^2} \hat{r}''$$

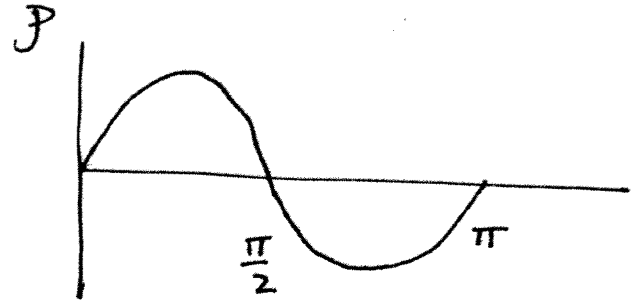
$\vec{r} = 0$ Field point

$\vec{r}' = r'\hat{r}'$ Source Point

$\vec{r}'' = \vec{r} - \vec{r}' = -r'\hat{r}'$ $\qquad r'' = r'$

$$\vec{E} = \frac{-1}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \int_0^{a} dr' \frac{\gamma \sin 2\theta'}{r'^2} r'^2 \sin\theta' \, \hat{r}'$$

$\hat{r}' = \cos\theta \, \hat{z}$ (Discarding $\hat{x}, \hat{y}$ which are zero by symmetry



Field should point in $\hat{z}$ direction

$$\vec{E} = -\frac{\gamma}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta' \int_0^{a} dr' \sin 2\theta' \sin\theta' \cos\theta' \,\hat{z}$$

$$= \frac{-2\pi a\gamma}{4\pi\varepsilon_0}\int_0^{\pi} d\theta' \sin 2\theta' \sin\theta' \cos\theta' \,\hat{z}$$

$$= \frac{-a\gamma}{4\varepsilon_0}\int_0^{\pi} \sin^2 2\theta' \, d\theta' \,\hat{z}$$

$$\cancel{\tfrac{4}{2}} \sin\theta \cos\theta = \frac{1}{2}\sin 2\theta$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \frac{\pi}{2}$$

$$\vec{E} = \frac{-a\pi\gamma}{4\varepsilon_0}\hat{z}$$

$$\int_0^{\pi} \sin^2 2\theta' d\theta' = \frac{1}{2}\int_0^{2\pi} \sin^2 u \, du$$

$$u = 2\theta' \quad du = 2d\theta'$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \frac{1}{2}\int_0^{2\pi} \sin^2\theta + \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta = \pi$$

$$\int_0^{\pi} \sin^2 2\theta \, d\theta = \frac{\pi}{2}$$

$$\vec{E} = \frac{-a\gamma}{4\epsilon_0} \frac{\pi}{2} \hat{z}$$

$$- \frac{a\pi\gamma}{8\epsilon_0} \hat{z}$$

Check Units

$$[\rho] = C/m^3 = [\gamma]$$

$$[E] = \left[\frac{C/m^2}{\epsilon_0}\right] \checkmark$$

Compute Potential at Origin Should be zero.

$$V(0) = \int \frac{\rho \, d\tau'}{4\pi\epsilon_0 r'}$$

$$= \int_0^a dr' \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \, \frac{r'^2 \sin\theta' \, \gamma \sin 2\theta'}{4\pi\epsilon_0 r'}$$

$$= \frac{\gamma}{4\pi\epsilon_0} \left[\int_0^{2\pi} d\phi'\right] \left[\int_0^a r' dr'\right] \left[\int_0^{\pi} d\theta' \sin\theta' \sin 2\theta'\right]$$

$$\underset{2\pi}{\shortparallel} \qquad \underset{a}{\shortparallel}$$

$$\int_0^{\pi} d\theta' \sin\theta' \sin 2\theta' = \int_0^{\pi} d\theta' \, 2 \sin^2\theta' \cos\theta'$$

$$u = \sin\theta' \qquad du = \cos\theta' \, d\theta'$$

$$= 2\int u^2 du = \frac{2}{3} u^3$$

$$= \frac{2}{3} \sin^3\theta \Big|_0^{\pi} = 0 \quad \checkmark$$

TOPS 35502

E.3.5

The electric field between the shells is

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

The energy density is then

$$u = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2$$

$$= \frac{Q^2}{32\pi^2\varepsilon_0 r^4}$$

Integrate the energy density over the volume between the spheres.

$$U = \int u\,dv = \int_a^b dr \int_0^{2\pi} r\sin\theta\, d\phi \int_0^{\pi} r\,d\theta\; u$$

$$= 4\pi\int_a^b r^2 dr\, u = 4\pi\left(\frac{Q^2}{32\pi^2\varepsilon_0}\right)\int_a^b \frac{dr}{r^2}$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left(-\frac{1}{r}\right)\Bigg|_a^b = \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

Could also get result from capacitance.

E (3.6) $V = V_0(x^2-y^2)$

$$= V_0(s^2\cos^2\phi - s^2\sin^2\phi)$$

$$= V_0 s^2(\cos^2\phi - \sin^2\phi)$$

$$= V_0 s^2 \cos 2\phi \qquad \text{(Trig identity wiki)}$$

<u>Electric field</u>

$$\vec{E} = -\nabla V = -V_0\left(\frac{\partial s^2\cos 2\phi}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial}{\partial\phi}s^2\cos 2\phi\,\hat{\phi}\right)$$

$$= -2V_0 s\cos 2\phi\,\hat{s} + 2V_0 s\sin 2\phi\,\hat{\phi} \qquad +5$$

<u>Charge Density</u> $\nabla\cdot\vec{E} = \rho/\varepsilon_0$

$$\rho = \varepsilon_0\nabla\cdot\vec{E}$$

$$= 2V_0\varepsilon_0\left[\frac{1}{s}\frac{\partial}{\partial s}\left(-s^2\cos 2\phi\right) + \frac{1}{s}\frac{\partial}{\partial\phi}\left(s\sin 2\phi\right)\right]$$

$$= 2V_0\varepsilon_0\left[-2\cos 2\phi + 2\cos 2\phi\right] = 0$$

+10

Try Cartesian

$$-\nabla V = \vec{E} = -2V_0(x\hat{x} - y\hat{y})$$

<u>Charge</u>

$$\varepsilon_0 \nabla \cdot \vec{E} = -2V_0\varepsilon_0(1-1) = 0$$