Homework 4

Due Monday 2/24/2014 - at beginning of class

Griffiths' 4 Problems

3.13 (Griffiths 3rd Edition 3.12)

3.19 (Griffiths 3rd Edition 3.18)

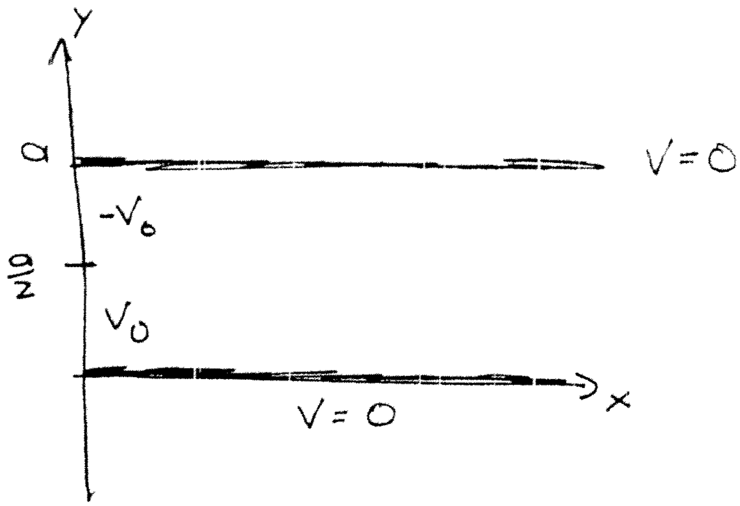
3.23 (Griffiths 3rd Edition 3.22) Work only up through P_3 not P_5.

Additional Problems

E.4.1 An infinite conducting cylinder of radius a is in an external electric field that is a uniform $E_0\hat{x}$ far from the cylinder. Compute the surface charge density as a function of ϕ on the surface of the cylinder.

E.4.2 The potential at the surface of an infinite cylinder of radius a is $V(a, \phi, z) = V_0 \cos(3\phi)$. Find the potential both inside and outside the cylinder. Find the field inside and outside and the surface charge density on the cylinder.

3.12



Two dimensional Laplace's Eqn

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Separate $V = X(x)\, Y(y)$

$$\underbrace{\frac{1}{X(x)} \frac{d^2 X}{dx^2}}_{= +k^2} + \underbrace{\frac{1}{Y} \frac{d^2 Y}{dy^2}}_{= -k^2} = 0$$

Y eqn

$$\frac{d^2Y}{dy^2} + k^2 Y = 0$$

$$Y = \sin kx, \quad \cos kx$$

For BC, $V(x,0,z) = 0$ $\quad V(x,a,z) = 0$

chose sine solution and let $k_n = \frac{n\pi}{a}$.

X eqn

$$\frac{d^2X}{dx^2} - k^2 X = 0$$

$$X = e^{kx}, \quad e^{-kx}$$

For the solution to be finite as $x \to \infty$,

$$X(x) = e^{-k_n x}$$

General Solution

$$V(x,y) = \sum_n A_n e^{-k_n x} \sin k_n y$$

Impose $x=0$ boundary condition

$$V(0,y) = \begin{cases} V_0 & 0<y<a/2 \\ -V_0 & a/2<y<a \end{cases}$$

$$= \sum_n A_n \sin k_n y$$

Multiply by $\sin k_m y$

$$V_0 \int_0^{a/2} \sin k_m y \, dy - V_0 \int_{a/2}^{a} \sin k_m y \, dy$$

$$= \sum_n A_n \underbrace{\int_0^a \sin k_m y \sin k_n y \, dy}_{\frac{a}{2}\delta_{nm}}$$

$$-\frac{V_0}{K_m}\cos K_m y\Big|_0^{a/2} + \frac{V_0}{K_m}\cos K_m y\Big|_{a/2}^{a}$$

$$= \frac{a}{2} A_m$$

$$= -\frac{V_0 a}{m\pi}\left(\cos\frac{m\pi}{2} - 1\right) + \frac{V_0 a}{m\pi}\left(\cos(m\pi) - \cos\frac{m\pi}{2}\right)$$

$$= \frac{V_0 a}{m\pi}\left(\cos m\pi - 2\cos\frac{m\pi}{2} + 1\right)$$

$$= \frac{4 V_0 a}{m\pi} \quad \text{if} \quad m = 2, 6, 10$$

which I simply read from the solution

0 otherwise

Let's work on it a bit

$$\cos\frac{m\pi}{2} = \pm\sqrt{\frac{1+\cos m\pi}{2}}$$

No luck, try a table

m	$\cos m\pi$	$-2\cos\frac{m\pi}{2}$	1	Total
1	-1	0	1	0
2	1	2	1	4
3	-1	0	1	0
4	1	-2	1	0
5	-1	0	1	0
6	1	2	1	4

So $\quad A_m = \dfrac{8V_0}{m\pi} \quad$ if $\quad m = 2, 6, 10, 14 \ldots$

$$V(x,y) = \frac{8V_0}{\pi} \sum_n \frac{1}{n} e^{-k_n x} \sin k_n y$$

3.19 $\quad V(R,\theta) = K\cos 3\theta$

Trig identity $\quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Legendre Polynomials

$P_0 = 1 \quad P_1 = \cos\theta \quad P_2 = \frac{1}{2}(3\cos^2\theta - 1)$

$P_3 = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

$$2P_3 = 5\cos^3\theta - 3P_1$$

$$\cos^3\theta = \frac{1}{5}(2P_3 + 3P_1)$$

$$\cos 3\theta = \frac{4}{5}(2P_3 + 3P_1) - 3P_1$$

$$= \frac{8}{5}P_3 - \frac{3}{5}P_1$$

$$V(R,0) = \frac{K}{5}(8P_3 - 3P_1)$$

Outside Discard r^n terms

$$V_o(r,\theta) = \sum_n A_n r^{-(n+1)} P_n(\cos\theta)$$

Let $a = R$

$$V_o(a,\theta) = \frac{8k}{5} P_3 - \frac{3k}{5} P_1 = \sum_n A_n^o a^{-(n+1)} P_n$$

$$\frac{A_1^o}{a^2} = -\frac{3k}{5} \qquad \frac{A_3^o}{a^4} = \frac{8k}{5}$$

$$V_o(r,\theta) = -\frac{3ka^2}{5r^2} P_1(\cos\theta) + \frac{8ka^4}{5r^4} P_3(\cos\theta)$$

Inside discard $r^{-(n+1)}$ terms

$$V_i(a,\theta) = \sum_n A_n^i r^n P_n(\cos\theta)$$

$$= \frac{8k}{5} P_3 - \frac{3k}{5} P_1$$

$$a^3 A_3^i = \frac{8k}{5} \qquad a A_1^i = \frac{-3k}{5}$$

$$V_i(r,\theta) = -\frac{3k}{5a} r P_1(\cos\theta) + \frac{8k}{5a^3} r^3 P_3(\cos\theta)$$

Surface Charge Density

$$\left.\frac{\partial V_o}{\partial r}\right|_a - \left.\frac{\partial V_i}{\partial r}\right|_a = \frac{-\sigma}{\varepsilon_0}$$

$$\left.\frac{\partial V_o}{\partial r}\right|_a = \frac{6ka^2}{5a^3} P_1 - \frac{32ka^4}{5a^5} P_3$$

$$\left.\frac{\partial V_i}{\partial r}\right|_a = \frac{-3k}{5a} P_1(\cos\theta) + \frac{24ka^2}{5a^3} P_3(\cos\theta)$$

$$\left.\frac{\partial V_o}{\partial r}\right|_a - \left.\frac{\partial V_i}{\partial r}\right|_a = \frac{k}{a}\left(\frac{9}{5} P_1 - \frac{56}{5} P_3\right) = \frac{-\sigma}{\varepsilon_0}$$

$$\sigma = \frac{k\varepsilon_0}{5a}\left(56 P_3 - 9 P_1\right)$$

3.23 Sphere with uniform surface charge density

$$\sigma(\theta) = \begin{cases} \sigma_0 & 0 \le \theta \le \pi/2 \\ -\sigma_0 & \frac{\pi}{2} < \theta < \pi \end{cases}$$

<u>Outside</u> (Discard r^n)

$$V_o(r,\theta) = \sum A_n^o \, r^{-(n+1)} P_n(\cos\theta)$$

$$\left.\frac{\partial V_o}{\partial r}\right|_a = \sum -(n+1) A_n^o \, a^{-(n+2)} P_n(\cos\theta)$$

<u>Inside</u> (Discard $r^{-(n+1)}$)

$$V_i(r,\theta) = \sum A_n^i \, r^n P_n(\cos\theta)$$

$$\left.\frac{\partial V_i}{\partial r}\right|_a = \sum_n n A_n^i \, r^{n-1} P_n(\cos\theta)$$

Potential Continuous at a

$$V_i(a,\theta) = V_o(a,\theta)$$

$$\sum_n A_n^i a^n P_n(\cos\theta) = \sum_n A_n^o a^{-(n+1)} P_n(\cos\theta)$$

By orthogonality, the series must be equal term by term

$$A_n^i a^n = A_n^o a^{-(n+1)}$$

$$A_n^o = a^{2n+1} A_n^i$$

Electrostatic Boundary Conditions at a

$$\left.\frac{\partial V_o}{\partial r}\right|_a - \left.\frac{\partial V_i}{\partial r}\right|_a = \frac{-\sigma}{\varepsilon_0}$$

$$\sum_n -(n+1) A_n^o a^{-(n+2)} P_n(\cos\theta) - \sum_n n A_n^i a^{n-1} P_n(\cos\theta) = -\frac{\sigma(x)}{\varepsilon_0}$$

$$-\sigma(x) = \sum_n P_n(\cos\theta)\Big[-A_n^o\, a^{-(n+2)}(n+1)$$
$$- n A_n^i\, a^{(n-1)}\Big]\varepsilon_0$$

$$= \sum_n A_n^i \varepsilon_0 P_n(\cos\theta)\Big[-(n+1)a^{-(n+2)}a^{2n+1}$$
$$- n\, a^{(n-1)}\Big]$$

$$= \sum_n \varepsilon_0 A_n^i P_n(\cos\theta)\, a^{(n-1)}\,[-2n-1]$$

$$\sigma(x) = \varepsilon_0 \sum_n (2n+1) A_n^i P_n(\cos\theta)\, a^{(n-1)}$$

<u>Multiply by P_m and integrate</u>

$$\int_{-1}^{1}\sigma(\theta)\, P_m(\cos\theta)\, d(\cos\theta) = \varepsilon_0 \sum_n (2n+1) a^{n-1} A_n^i \,\times$$

$$\underbrace{\int_{-1}^{1} P_m(\cos\theta) P_n(\cos\theta)\, d\cos\theta}_{\text{orthogonality condition}}$$

$$\parallel$$

$$\frac{2}{2m+1}\,\delta_{nm}$$

$$\int_{-1}^{1} \sigma(\theta)\, P_m(\cos\theta)\, d(\cos\theta) = \epsilon_0 \sum_n (2n+1) A_n^i\, a^{n-1} \cdot \frac{2}{2m+1}\, \delta_{nm}$$

$$= 2\,\epsilon_0\, A_m^i\, a^{m-1}$$

<u>Work on the integral</u>

$$\int_{-1}^{1} \sigma(\theta)\, P_m(\cos\theta)\, d(\cos\theta) = -\int_{\pi}^{0} \sigma(\theta)\, P_m(\cos\theta) \sin\theta\, d\theta$$

$$= \int_0^{\pi} \sigma(\theta) \sin\theta\, P_m(\cos\theta)\, d\theta \equiv I_m$$

<u>Boundary Condition</u>

$$\sigma(\theta) = \begin{cases} \sigma_0 & 0 \leq \theta \leq \pi/2 \\ -\sigma_0 & \pi/2 \leq \theta \leq \pi \end{cases}$$

$$I_m = \sigma_0 \int_0^{\pi/2} P_m(\cos\theta) \sin\theta\, d\theta - \sigma_0 \int_{\pi/2}^{\pi} P_m(\cos\theta)\, d(\cos\theta)$$

$$u = \cos\theta \qquad du = -\sin\theta\, d\theta$$

$$I_m = -\sigma_0 \int_{-1}^{0} P_m(u)\,du + \sigma_0 \int_0^1 P_m(u)\,du$$

$$= \sigma_0 \int_0^1 P_m(u)\,du - \sigma_0 \int_{-1}^{0} P_m(u)\,du$$

Evidenty $I_m = 0$ for even functions, and

$$I_m = 2\sigma_0 \int_0^1 P_m(u)\,du$$

for odd functions.

~~P_0 even $I_0 = 2\sigma_0 \int_0^1 P_0\,du = 2\sigma_0 \int_0^1 du = 2\sigma_0$~~

$P_0 = 1$, $I_0 = 0$ since even.

$P_1 = x$, $I_1 = 2\sigma_0 \int_0^1 x\,dx = \sigma_0$

~~$P_2 = 0$~~ 3, $P_2 = \frac{1}{2}(3x^2 - 1)$, $I_2 = 0$ since even.

$P_3 = \frac{1}{2}(5x^3 - 3x)$ $\quad I_3 = \sigma_0 \int_0^1 (5x^3 - 3x)\,dx$

$$= \sigma_0\left(\frac{5}{4} - \frac{3}{2}\right) = \frac{-\sigma_0}{4}$$

$I_0 = 0 \implies \boxed{A_0 = 0}$

$I_1 = \sigma_0 = 2\varepsilon_0 A_1^i a^{1-1}$

$$\boxed{A_1^i = \frac{\sigma_0}{2\varepsilon_0}}$$

$I_2 = 0 \implies \boxed{A_2 = 0}$

$I_3 = -\frac{\sigma_0}{4} = 2\varepsilon_0 A_3^i a^{3-1}$

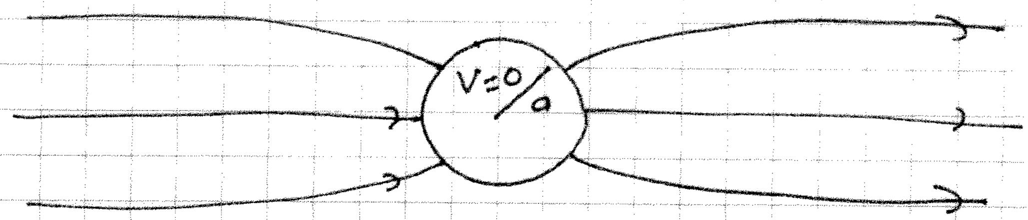
$$\boxed{A_3^i = \frac{-\sigma_0}{8\varepsilon_0 a^2}}$$

Outside

$A_0^o = 0 \qquad A_1^o = a^3 A_1^i = \frac{\sigma_0 a^3}{2\varepsilon_0}$

$A_2^o = 0 \qquad A_3^o = a^7 A_3^i = \frac{-\sigma_0 a^5}{8\varepsilon_0}$

E.4.1



$$\vec{E} = E_0 \hat{x} \quad \text{as} \quad x \to \pm\infty$$

$$V = -E_0 x = -E_0 s \cos\phi$$

The general solution is

$$V(s,\phi) = \sum_n A_n s^{-n} \cos n\phi + B_n s^{-n} \sin n\phi + C_n s^n \cos n\phi + D_n s^n \sin n\phi$$

As $s \to \infty$, $V \to -E_0 s \cos\phi$ so

$C_1 = -E_0$, $C_0, C_{n\geq 2} = 0$ and $D_n = 0$

$$V(s,\phi) = \sum_n A_n s^{-n} \cos n\phi + B_n s^{-n} \sin n\phi - E_0 s \cos\phi$$

Apply Boundary Condition

$$V(a,\phi) = 0 = \sum A_n a^{-n} \cos n\phi + B_n a^{-n} \sin n\phi$$
$$- E_0 a \cos\phi$$

By orthogonality, $A_n = 0 \quad n \neq 1, \quad B_n = 0$

$$A_1 a^{-1} = +E_0 a$$

$$A_1 = +a^2 E_0$$

$$V(s,\phi) = -E_0 s \cos\phi + \frac{E_0 a^2}{s} \cos\phi$$

Electric Field

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial s}\hat{s} - \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\phi}$$

$$= \left(E_0 \cos\phi + \frac{E_0 a^2}{s^2}\cos\phi\right)\hat{s}$$

$$\frac{1}{s}\left(-E_0 s + \frac{E_0}{s} a^2\right)\sin\phi\,\hat{\phi}$$

At $s = a$, $\quad \vec{E} = 2E_0 \cos\phi\,\hat{s}$

Charge Density (Gaussian Pillbox)

$$\phi = |E|A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\sigma = \varepsilon_0 |E| = 2\varepsilon_0 E_0 \cos\phi$$

E.4.2

Outside

$$V_o(s,\phi) = \sum_n A_n s^{-n} \cos n\phi + B_n s^{-n} \sin n\phi$$

By orthogonality, only $\cos 3\phi$ survives.

$$V_o(s,\phi) = \frac{A_3}{s^3} \cos 3\phi$$

B.C

$$V(a,\phi) = V_0 \cos 3\phi = \frac{A_3}{a^3} \cos 3\phi$$

$$A_3 = V_0 a^3$$

$$V_o(s,\phi) = \frac{V_0 a^3}{s^3} \cos 3\phi$$

Inside

$$V_i(s,\phi) = \sum_n C_n s^n \cos n\phi + D_n s^n \sin n\phi$$

by orthogonality

$$V_i(s,\phi) = C_3 s^3 \cos 3\phi$$

Boundary Conditions

$$V(a,\phi) = V_0 \cos 3\phi = C_3 a^3 \cos 3\phi$$

$$C_3 = V_0 / a^3$$

$$V_i(s,\phi) = \frac{V_0 s^3}{a^3} \cos 3\phi$$

Field

$$\nabla f = \hat{s}\frac{\partial f}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial f}{\partial \phi}$$

Field Inside

$$\vec{E}_i = -\nabla V_i = -\left(\frac{3V_0 s^2}{a^3}\cos 3\phi\,\hat{s} - \frac{3V_0 s^2}{a^3}\sin 3\phi\,\hat{\phi}\right)$$

Field Outside

$$\vec{E}_o = -\nabla V_o = -\left(-\frac{3V_0 a^3}{s^4}\cos 3\phi\,\hat{s} - \frac{3V_0 a^3}{s^4}\sin 3\phi\,\hat{\phi}\right)$$

or re-writing

$$\vec{E}_i = \frac{3V_0 s^2}{a^3}\left(-\cos 3\phi\,\hat{s} + \sin 3\phi\,\hat{\phi}\right)$$

$$\vec{E}_o = \frac{3V_0 a^3}{s^4}\left(\cos 3\phi\,\hat{s} + \sin 3\phi\,\hat{\phi}\right)$$

Surface Charge   Use Gaussian pillbox at surface

$$\sigma = \varepsilon_0 \hat{n} \cdot \left(\vec{E}_o - \vec{E}_i \right) \qquad \hat{n} = \hat{s}$$

$$= \varepsilon_0 \left(\frac{3 V_0 a^3}{a^4} \cos 3\phi - \frac{3 V_0 a^2}{a^3} \left(-\cos 3\phi \right) \right)$$

$$= \frac{6 V_0 \varepsilon_0}{a} \cos 3\phi$$