# Homework 5

Due Friday 3/10/2014 - at beginning of class

## Griffiths' 4 Problems

**3.9** (Griffiths 3rd Edition 3.8)

**3.10** (Griffiths 3rd Edition 3.9)

**3.30** (Griffiths 3rd Edition 3.28)

## Additional Problems

**E.5.1** An infinite grounded conducting plane occupies the area $z < 0$. A dipole is formed of two charges $\pm Q$ separated by a distance $a$. The dipole moment is parallel to the $z$ axis. The center or the dipole is a distance $R$ from the plane. Compute the force on each charge and the total force on the dipole. Is the dipole attracted or repelled from the plane?

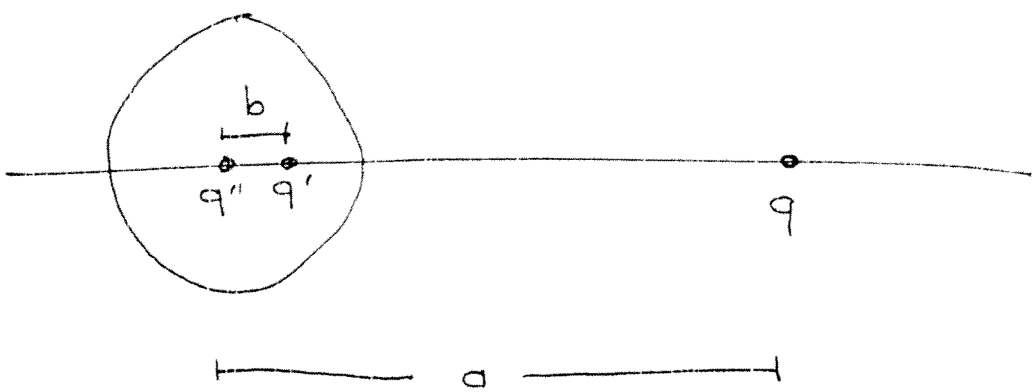
**E.5.2** A grounded conducting sphere of radius $a$ is centered at the origin. Two charges $+q$ are at a location $\pm D\hat{x}$ along the $x$-axis. Compute the electric potential at the point $2D\hat{x}$.

**E.5.3** The potential on the surface of an infinite cylinder is given by $V(a,\phi) = V_0(\sin(\phi) + \cos(\phi))$. Find the potential at all points inside the cylinder and the field inside the cylinder.

**E.5.4** Find the potential in the region where $x > 0$ and $y > 0$. The $y - z$ plane is held at potential $V_0$ and the $x - z$ plane is grounded. Hint, look at the trivial solutions.

**E.5.5** The potential of a rectangular system is independent of $z$. The system extends to infinity in the $x$ direction. On the plane $y = 0$ and $y = a$ the potential is zero. On the plane $x = 0$, the potential is $V(0, y) = V_0 \sin^2(\pi y/a)$. Compute the potential in the channel. Report the integral you would use to calculate the coefficients in the series, but you do not have to evaluate the integral.

3.9 Force of attraction point charge $q$ and neutral conducting sphere.




To bring the surface of the sphere to zero potential use an image charge

$$q' = -\frac{R}{a} q$$

$$\text{at } b = \frac{R^2}{a}$$

This leaves the sphere with charge $q'$. To return the sphere to $Q=0$. Place a charge $q'' = 0 - q'$ at the origin.

$$F = \frac{Kqq''}{a^2} + \frac{Kqq'}{(a-b)^2}$$

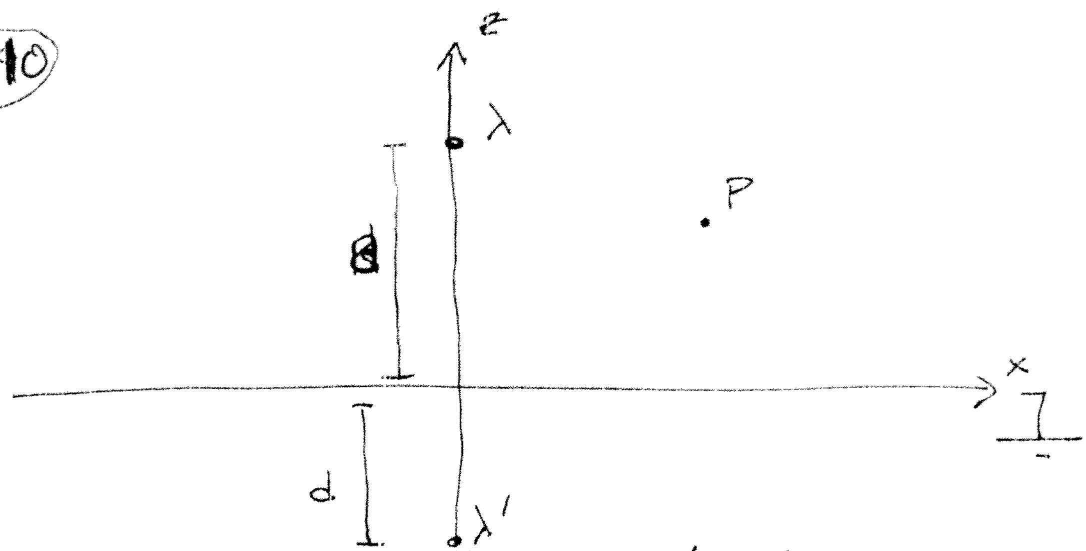
$$= Kqq' \left( \frac{1}{(a-b)^2} - \frac{1}{a^2} \right)$$

$$= -\frac{Kq^2R}{a} \left( \frac{1}{(a - R^2/a)^2} - \frac{1}{a^2} \right)$$

Now the first part of the problem. To leave the surface at $V_0$ place a charge $q''$ in the center. Choose $q''$ so that

$$V_0 = \frac{Kq''}{R}$$

3.10




Sln Use image line charge, $\lambda' = -\lambda$ at a distance d below the plane.

The field of an infinite line charge is

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

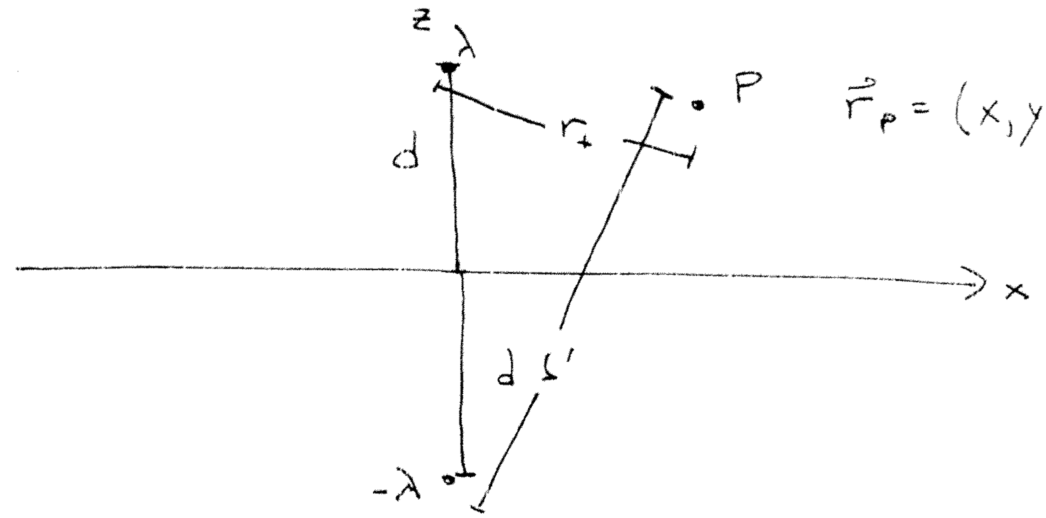
so the potential is

$$V = -\int E\,dr = -\frac{\lambda}{2\pi\varepsilon_0}\ln(r) + C$$

Make the choice, $V(0,0,0) = 0 \Rightarrow -\frac{\lambda}{2\pi\varepsilon_0}\ln(d) + C = 0$

$$C = \frac{\lambda}{2\pi\varepsilon_0}\ln(d)$$

$$V(r) = -\frac{\lambda}{2\pi\varepsilon_0}\ln(r/d)$$




$\vec{r}_p = (x, y, z)$

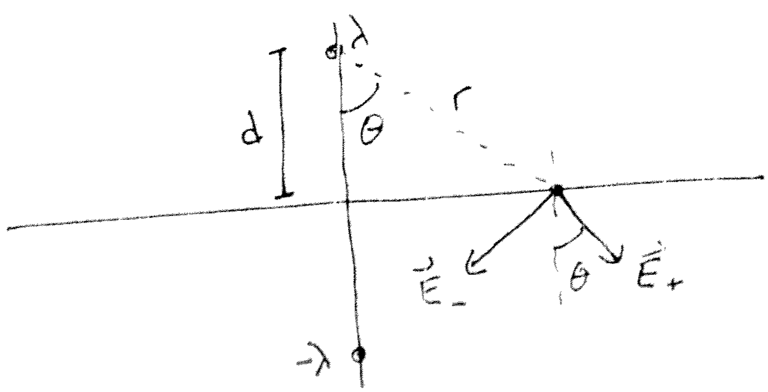
$$r_+ = \sqrt{x^2 + (z-d)^2} \qquad r_- = \sqrt{x^2 + (z+d)^2}$$

$$V_p = -\frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_+}{d}\right) + \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_-}{d}\right)$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_-}{r_+}\right) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{\sqrt{x^2+(z+d)^2}}{\sqrt{x^2+(z-d)^2}}\right)$$

---

(b)
The electric field at a point on the surface



The x, y components of the field cancel

$$\vec{E}_{plane} = \vec{E}_+ + \vec{E}_- = 2\,|E_+| \cos\theta$$

$$= \frac{2\lambda}{2\pi\varepsilon_0 r} \cdot \frac{d}{r}$$

$$= \frac{\lambda d}{\pi\varepsilon_0} \frac{1}{r^2}$$

Use Gaussian pillbox at surface

$$\Phi = \vec{E}\cdot\hat{n}\,A = \frac{\lambda d}{\pi\varepsilon_0}\frac{1}{r^2}A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\sigma = \frac{-\lambda d}{\pi r^2} = \frac{-\lambda d}{\pi\left(x_i^2 + d^2\right)}$$

Note, $\sigma$ must be negative

3.38

$$\vec{p} = \int \vec{r}\,\sigma(\vec{r})\,da \qquad da = R\,d\theta\,R\sin\theta\,d\phi$$

$$p_x = p_y = 0 \quad \text{by symmetry}$$

$$p_z = \int z\,\sigma(\vec{r})\,R\,d\theta\,R\sin\theta\,d\phi$$

$$z = r\cos\theta = R\cos\theta$$

$$p_z = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\,\underbrace{(R\cos\theta)}_{z=}\underbrace{(k\cos\theta)}_{\sigma}\underbrace{R^2\sin\theta}_{da}$$

$$p_z = 2\pi k R^3 \int_0^{\pi} \cos^2\theta\,\sin\theta\,d\theta$$

Let $u = \cos\theta \qquad du = -\sin\theta\,d\theta$

$$p_z = -2\pi k R^3 \int_1^{-1} u^2\,du = 2\pi k R^3 \left.\frac{u^3}{3}\right|_{-1}^{1}$$

$$= \frac{4}{3}\pi k R^3$$

$$\vec{p} = \frac{4}{3}\pi k R^3 \hat{z}$$

(b) The dipole potential is

$$V_{dip} = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$$

$$= \frac{|p| \cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\frac{4}{3}\pi k R^3 \cos\theta}{4\pi\varepsilon_0 r^2}$$

$$= \frac{kR^3}{3\varepsilon_0 r^2}\cos\theta$$

which is the exact potential, so there are no higher order multipoles.

## E.5.1

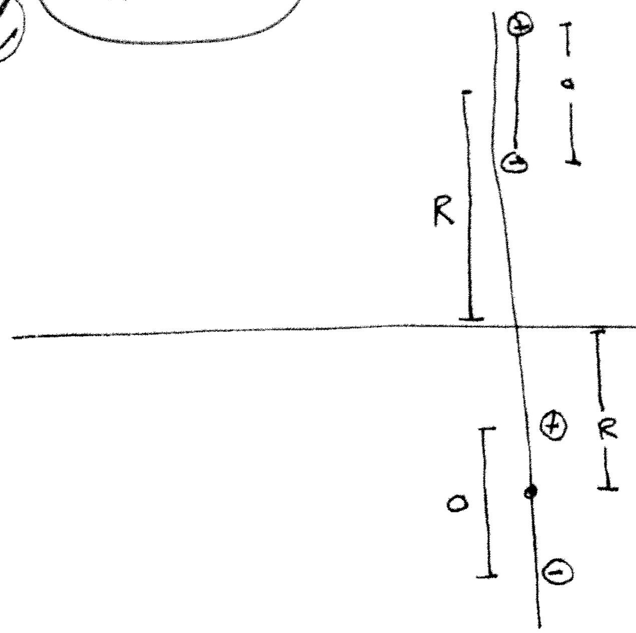



Image Dipole - Charges equal

The force exerted on the negative charge is

$$F_- = \frac{-kq^2}{(2R-a)^2} + \frac{kq^2}{(2R)^2}$$

] upward positive

and the force on the positive charge

$$F_+ = \frac{kq^2}{(2R)^2} - \frac{kq^2}{(2R+a)^2}$$

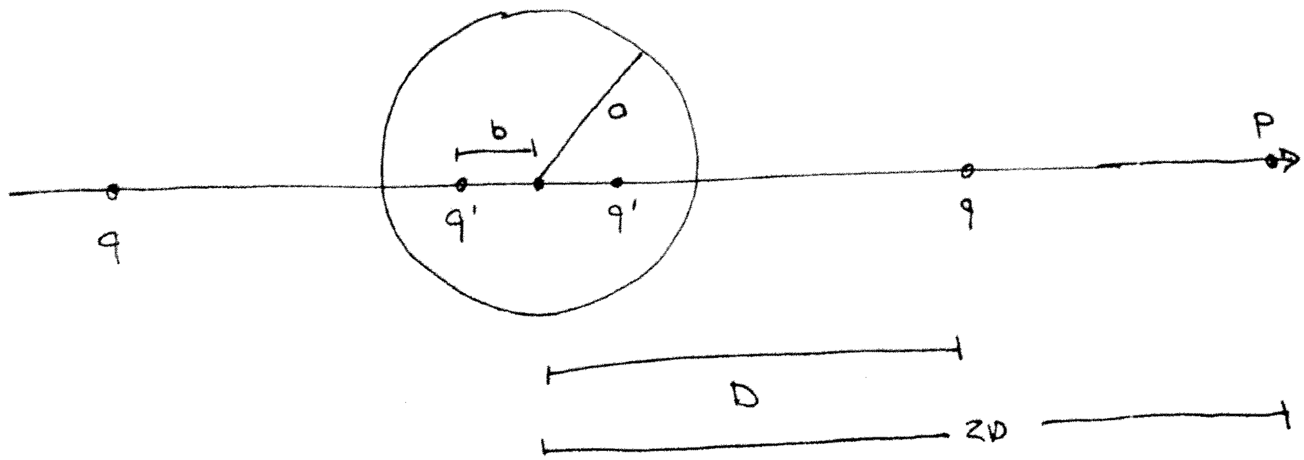
The total force exerted by the plane on the dipole is

$$F = F_- + F_+ = 2 \cdot \frac{kq^2}{4R^2} - \frac{kq^2}{(2R+a)^2} - \frac{kq^2}{(2R-a)^2}$$

$$= \frac{kq^2a^2(a^2-12R^2)}{2R^2(2R+a)^2(2R-a)^2} < 0$$

You could also argue the force was attractive because the dipoles are anti-aligned.

## E.5.2




To bring the sphere to zero potential use two image charges

$$q' = -\frac{a}{D} q$$

at $\qquad \pm b = \frac{a^2}{D}$

The total potential at P is the sum of the potentials.

$$V = \frac{kq}{3D} + \frac{kq'}{(b+2D)} + \frac{kq'}{(2D-b)} + \frac{kq}{D}$$

$$= \frac{4kq(a^4 - 4D^4 + 3aD^3)}{3D(a^4 - 4D^4)}$$

If you simplified, Maple says...

E.5.3 Cylindrical System $s<a$

$$V(s,\phi) = \sum_n A_n s^n \sin n\phi + B_n s^n \cos n\phi$$

discarding terms that blow up at the origin.

Apply B.C.

$$V(a,\phi) = V_0 \sin\phi + V_0 \cos\phi$$

$$= \sum_n A_n a^n \sin n\phi + B_n a^n \cos n\phi$$

By orthogonality,

$$A_n = B_n = 0 \qquad n \neq 1$$

$$A_1 a = V_0 \qquad B_1 a = V_0$$

$$V(s,\phi) = \frac{V_0}{a} s \sin\phi + \frac{V_0}{a} s \cos\phi$$

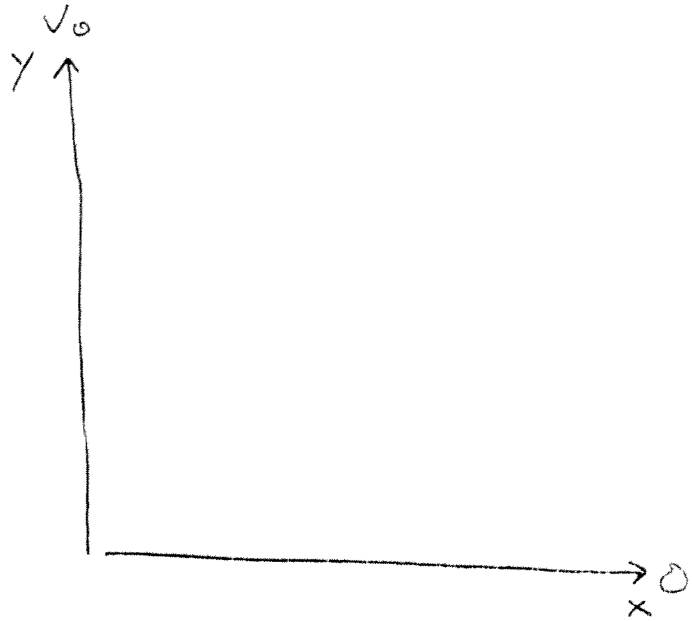
## Compute Field

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial s}\hat{s} - \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\phi}$$

$$= -\frac{V_0}{a}(\sin\phi + \cos\phi)\,\hat{s}$$

$$-\frac{V_0}{a}(\cos\phi - \sin\phi)\,\hat{\phi}$$

E.5.4




Trivial Solution $V = C\phi + D$ satisfies B.C.

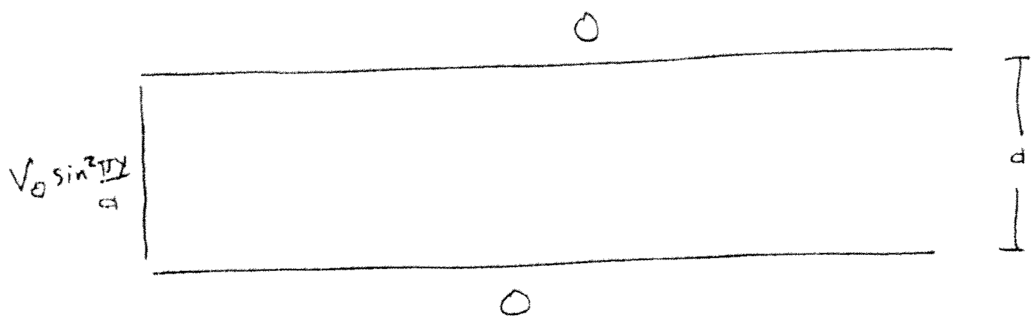
so by uniqueness it is the solution.

$$V(0) = D = 0$$

$$V\left(\frac{\pi}{2}\right) = C\frac{\pi}{2} = V_0 \implies C = \frac{2V_0}{\pi}$$

$$V(s,\phi,z) = \frac{2V_0}{\pi}\phi$$

~~E.4.4~~ E.5.5




Separated Laplacian

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$k^2 \qquad\qquad -k^2$$

The solutions are $[e^{kx}, e^{-kx}] \times [\sin ky, \cos ky]$

- To meet the y boundary condition, discard $\cos ky$ and set $k = \frac{n\pi}{a}$
- Discard $e^{kx}$ because it blows up at $\infty$.

The general solution is then

$$V(x,y) = \sum A_n e^{-k_n x} \sin k_n y$$

At $x = 0$

$$V(0,y) = V_0 \sin^2 \frac{\pi y}{a} = \sum A_n \sin k_n y$$

Use orthogonality $\int_0^a \sin k_n y \sin k_m y \, dy = \frac{a}{2} \delta_{nm}$

$$\frac{a}{2} A_m = V_0 \int_0^a \sin \frac{m\pi}{b} y \sin^2 \frac{\pi y}{a} \, dy$$

$$A_m = \frac{2V_0}{a} \int_0^a \sin \frac{m\pi}{b} y \sin^2 \frac{\pi y}{a} \, dy$$