Homework 6

Due Monday 3/17/2014 - at beginning of class

Griffiths' 4 Problems (3rd Edition numbers are the same)

4.5

4.15

4.20

4.21

Additional Problems

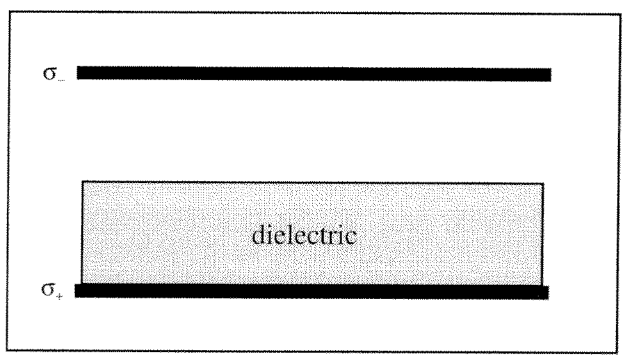
E.6.1 A dipole formed of a $+Q$ and $-Q$ charge spaced a distance a apart has dipole moment pointing in the $+\hat{z}$ direction. The center of the dipole is located at $+R\hat{z}$ a distance R above a neutral dielectric slab occupying the volume $z < 0$ with dielectric constant κ. Compute the force the dielectric plane exerts on the dipole.

E.6.2 A point charge $+Q$ is a distance R above a neutral dielectric slab with dielectric constant κ occupying the volume $z < 0$. Compute the electric field immediately above and below the dielectric surface. From the field, calculate the bound charge density at the surface.

E.6.3 A linear dielectric with dielectric constant κ occupies the volume $-a < z < a$. A uniform volume charge density ρ is fixed within the dielectric. Compute the electric field everywhere. Compute the polarization everywhere.

E.6.4 A spherical system has polarization $\vec{P} = \gamma r^2 \hat{r}$ for radius $r < a$ and $\vec{P} = 0$ for $r > a$. Find the electric field everywhere.

E.6.5 A linear dielectric slab with dielectric constant ϵ_r is placed between two infinite parallel planes of charge with charge density $\pm\sigma$. Find $\vec{D}$, $\vec{E}$, $\vec{P}$, and ρ_b in the dielectric, and the bound charge density on the top and bottom surface of the dielectric.



E.6.6 A potential of $V_0 \cos(\theta)$ is established on the inner surface of a spherical dielectric with inner radius a and outer radius b. The dielectric constant of the material is ϵ_r. Find the potential for $r > a$. You may report a system of equations that needs to be solved to find the coefficients of the potential functions. Actually solving these equations turns out to be quite messy. These equations should be a set of simple linear, non-differential equations.

4.5 Torque on dipole $\vec{\tau} = \vec{p} \times \vec{E}$

Electric field of dipole, $\vec{p} = p\hat{y}$, along axis.

<u>x-axis</u> $\vec{E} = -\dfrac{kp}{x^3}\hat{y}$ <u>y-axis</u> $\vec{E} = \dfrac{2kp}{y^3}\hat{y}$

<u>Torque on p_1 due to p_2</u>

$$\vec{E}_{21} = \frac{2kp_2}{r^3}\hat{x} \qquad \vec{p}_1 = p_1\hat{y}$$

$$\vec{\tau}_{21} = \vec{p}_1 \times \vec{E}_{21} = (p_1\hat{y}) \times \left(\frac{2kp_2}{r^3}\hat{x}\right)$$

$$= -\frac{2kp_1p_2}{r^3}\hat{z}$$

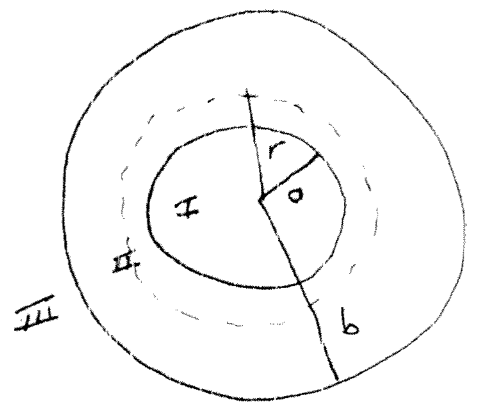
<u>Torque on p_2 due to p_1</u>

$$\vec{E}_{12} = -\frac{kp_1}{r^3}\hat{y} \qquad \vec{p}_2 = p_2\hat{x}$$

$$\vec{\tau}_{12} = \vec{p}_2 \times \vec{E}_{12} = (p_2\hat{x}) \times \left(-\frac{kp_1}{r^3}\hat{y}\right)$$

$$= -\frac{kp_1p_2}{r^3}\hat{z}$$

4.15



$$\vec{P} = \frac{k}{r}\hat{r}$$

(a) Bound charge

$$\sigma_b = \vec{P}\cdot\hat{n}$$

<u>Inner Surface</u> $\hat{n} = -\hat{r}$

$$\sigma_b = -\frac{k}{a} \qquad Q_{\cancel{outer}\ inner} = 4\pi a^2 \sigma_b = -4\pi k a$$

<u>Outer Surface</u> $\hat{n} = \hat{r}$

$$\sigma_b = \frac{k}{b} \qquad Q_{outer} = 4\pi b^2 \sigma_b = 4\pi k b$$

<u>Volume Charge</u>

$$\rho_b = -\nabla\cdot\vec{P} = -k\,\nabla\cdot\left(\frac{\hat{r}}{r}\right)$$

$$= \frac{-k}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{1}{r}\right) = -\frac{k}{r^2}$$

<u>Region III</u> $\quad Q_{enc} = Q_{inner} + Q_{vol}(b) + Q_{outer}$

$$= -4\pi k a - 4\pi k (b-a) + 4\pi k b$$

$$= 0$$

$$\vec{E}_{III} = 0$$

(b) There is no free charge,

$$\nabla \cdot \vec{D} = 0 \quad \Longrightarrow \quad \vec{D} = 0$$

(constant solution violates spherical symmetry)

In region I, III $\quad \vec{P} = 0$

$$\vec{D} = 0 = \varepsilon_0 \vec{E} + \vec{P} \Rightarrow \vec{E} = 0$$

$$\vec{E}_I = 0 \qquad\qquad \vec{E}_{III} = 0$$

In region II,

$$\vec{D} = 0 = \varepsilon_0 \vec{E}_{II} + \vec{P}$$

$$\vec{E}_{II} = -\frac{\vec{P}}{\varepsilon_0} = -\frac{k}{\varepsilon_0 r}\hat{r}$$

Gauss Law

Region I $\quad r < a \quad Q_{enc} = 0 \quad \vec{E}_I = 0$

Region II $\quad a < r < b \quad Q_{enc} = Q_{inner} + Q_{vol}(r)$

$$Q_{vol}(r) = \int_a^r 4\pi r^2 \rho_o \, dr$$

$$= \int_a^r 4\pi r^2 \left(-\frac{k}{r^2}\right) dr$$

$$= -4\pi k(r-a)$$

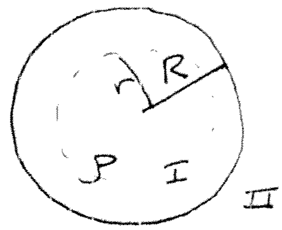
$$Q_{enc} = -4\pi k a - 4\pi k(r-a) = -4\pi k r$$

Gauss $\quad \Phi = 4\pi r^2 E = Q_{enc}/\varepsilon_0$

$$\vec{E}_{II} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{-4\pi k r}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$\vec{E}_{II} = \frac{-k}{\varepsilon_0 r}\hat{r}$$

4.20



<u>Gauss Law</u>

<u>Region I</u> $r < R$

$$Q_{enc} = \frac{4}{3}\pi r^3 \rho$$

$$\Phi = 4\pi r^2 D = Q_{enc}$$

$$\vec{D}_I = \frac{\frac{4}{3}\pi r^3 \rho}{4\pi r^2}\hat{r} = \frac{\rho r}{3}\hat{r}$$

$$\vec{D}_I = \varepsilon_r \varepsilon_0 \vec{E}_I$$

$$\vec{E}_I = \frac{\rho r}{3\varepsilon_r \varepsilon_0}\hat{r} = \frac{\rho r}{3\varepsilon_r \varepsilon_0}\hat{r}$$

Region II $\quad r > R$

$$Q_{fenc} = \frac{4}{3}\pi R^3 \rho$$

$$\vec{D}_{II} = \frac{Q_{fenc}}{4\pi r^2}\hat{r} = \frac{\frac{4}{3}\pi R^3 \rho}{4\pi r^2}\hat{r}$$

$$= \frac{R^3 \rho}{3 r^2}\hat{r}$$

$$\vec{D}_{II} = \varepsilon_0 \vec{E}_{II}$$

$$\vec{E}_{II} = \frac{R^3 \rho}{3 \varepsilon_0 r^2}\hat{r}$$

Potential Difference

$$\Delta V_{\infty 0} = -\int_{\infty}^{0} \vec{E} \cdot d\vec{l} = -\int_{\infty}^{R} \vec{E}_{II} \cdot d\vec{l} - \int_{R}^{0} \vec{E}_{I} \cdot d\vec{l}$$

$$d\vec{l} = dr\,\hat{r} \quad (dr < 0 \text{ because of limits})$$

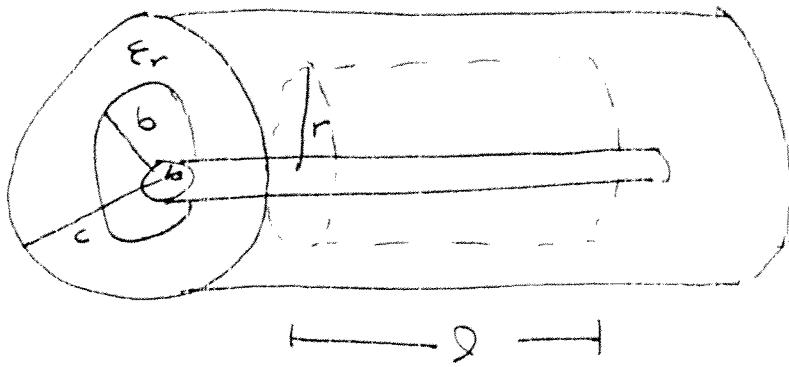
$$\Delta V_{\infty 0} = -\int_{\infty}^{R} \frac{R^3 \rho}{3\varepsilon_0 r^2}\,dr - \int_{R}^{0} \frac{\rho r}{3\varepsilon_r \varepsilon_0}\,dr$$

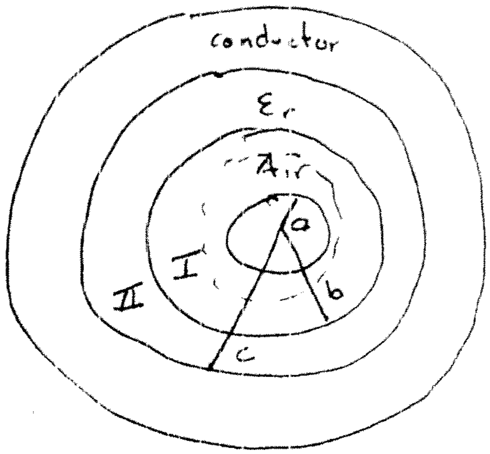
$$= \left.\frac{R^3 \rho}{3\varepsilon_0 r}\right|_{\infty}^{R} - \left.\frac{\rho r^2}{6\varepsilon_r \varepsilon_0}\right|_{R}^{0}$$

$$\Delta V_{\infty 0} = \frac{R^3 \rho}{3 \varepsilon_0} \left(\frac{1}{R} - \frac{1}{\infty} \right) + \frac{\rho R^2}{6 \varepsilon_0 \varepsilon_r}$$

$$= \frac{\rho R^2}{3 \varepsilon_0} \left(1 + \frac{1}{2 \varepsilon_r} \right)$$

4.21





Add +Q charge per length l of the Gaussian surface.

Q_{fenc} in cylindrical Gaussian surface of radius r

is $Q_{fenc} = Q$

<u>Gauss Law</u> $\Phi_d = 2\pi r l D = Q_{fenc}$

$$\vec{D} = \frac{Q}{l} \cdot \frac{1}{2\pi r}$$

Region I $\quad a < r < b$

$$\vec{D} = \varepsilon_0 \vec{E}_I$$

$$\vec{E}_I = \frac{Q}{\ell} \cdot \frac{1}{2\pi\varepsilon_0 r} \hat{r}$$

Note, using $\hat{r} = \hat{s}$ because of habit. Definitely a cylindrical problem.

Region II $\quad b < r < c$

$$\vec{D} = \varepsilon_r \varepsilon_0 \vec{E}_{II}$$

$$\vec{E}_{II} = \frac{Q}{\ell} \frac{1}{2\pi\varepsilon_0 \varepsilon_r r} \hat{r}$$

Potential Difference

$$|\Delta V| = |\Delta V_I| + |\Delta V_{II}| \quad \text{(fields in same direction)}$$

$$|\Delta V_I| = \left| - \int_a^b \vec{E}_I \cdot d\vec{\ell} \right| \qquad d\vec{\ell} = \hat{r}\, dr$$

$$= \left| - \frac{Q}{\ell} \cdot \frac{1}{2\pi\varepsilon_0} \int_a^b \frac{dr}{r} \right| = \frac{Q}{\ell} \cdot \frac{1}{2\pi\varepsilon_0} \ln\left(b/a\right)$$

$$|\Delta V_{II}| = \left| -\int_b^c \vec{E}_{II} \cdot d\vec{l} \right| = \left| -\frac{Q}{l} \frac{1}{2\pi\varepsilon_0\varepsilon_r} \int_b^c \frac{dr}{r} \right|$$

$$= \frac{Q}{l} \frac{1}{2\pi\varepsilon_0\varepsilon_r} \ln(c/b)$$

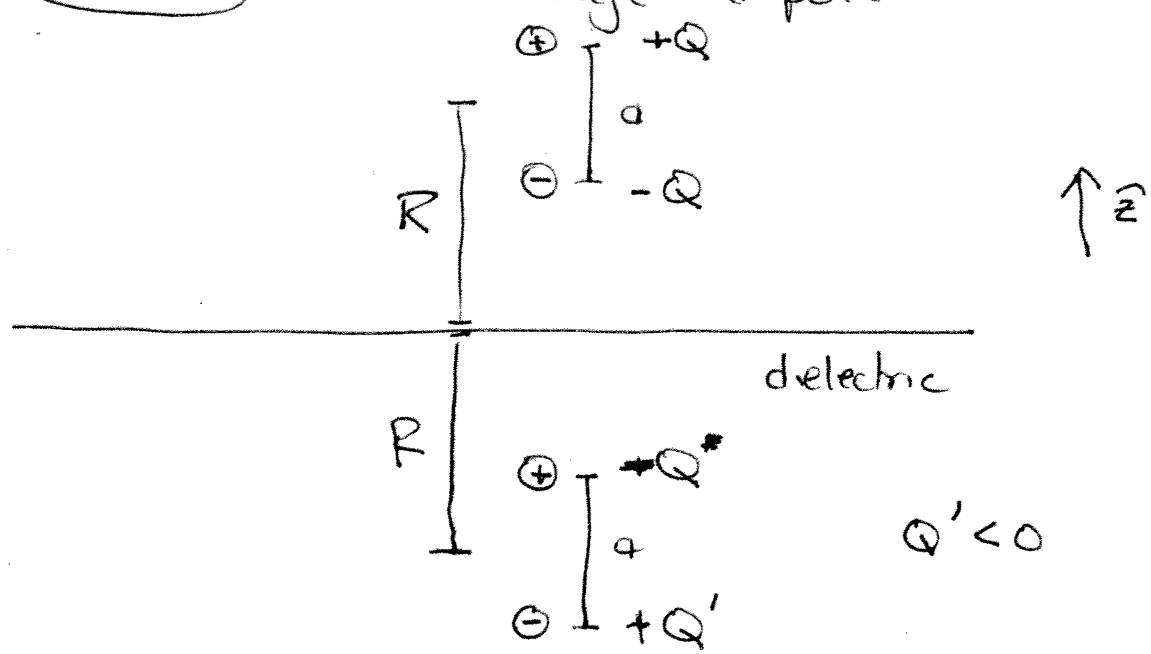
$$|\Delta V| = |\Delta V_I| + |\Delta V_{II}|$$

$$= \frac{Q}{2\pi\varepsilon_0 l}\left(\ln(b/a) + \frac{1}{\varepsilon_r}\ln(c/b) \right)$$

$$C = \frac{Q}{|\Delta V|} = \frac{2\pi\varepsilon_0 l}{\ln(b/a) + \frac{1}{\varepsilon_r}\ln(c/b)}$$

$$\frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln(b/a) + \frac{1}{\varepsilon_r}\ln(c/b)}$$

E.6.1 Use image dipole



$Q' < 0$

$$Q' = -\left(\frac{\kappa - 1}{\kappa + 1}\right) Q = -\frac{\chi_e}{\chi_e + 2} Q \equiv -\gamma Q$$

Force

$$\vec{F}_{dip} = \frac{-kQQ'}{(2R)^2} - \frac{kQQ'}{(2R)^2} + \frac{kQQ'}{(2R+a)^2} + k\frac{QQ'}{(2R-a)^2}$$

$$= k\gamma Q^2 \left[\frac{1}{4R^2} + \frac{1}{4R^2} - \frac{1}{(2R+a)^2} - \frac{1}{(2R-a)^2}\right] \times \hat{z}$$

$$= -k\gamma Q^2 \frac{(12a^2R^2 - a^4)}{2R^2(2R-a)^2(2R+a)^2}$$

(alpha)

WolframAlpha

simplify 1/(2*r^2)-1/(2r-a)^2-1/(2r+a)^2

Examples Random

Input interpretation:

simplify $\dfrac{1}{2r^2} - \dfrac{1}{(2r-a)^2} - \dfrac{1}{(2r+a)^2}$

Results: More

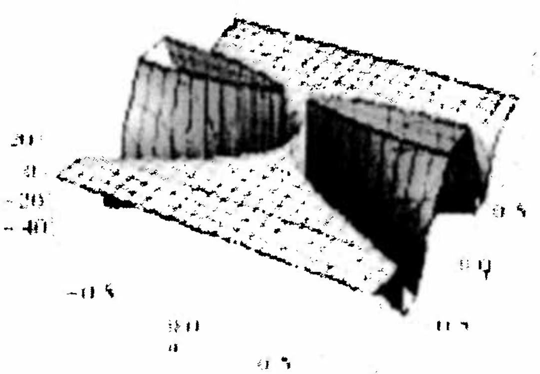
$$-\frac{1}{(a+2r)^2} - \frac{1}{(a-2r)^2} + \frac{1}{2r^2}$$

$$\frac{a^4 - 12a^2r^2}{2(a^2r - 4r^3)^2}$$

$$-\frac{12a^2r^2 - a^4}{2r^2(a-2r)^2(a+2r)^2}$$

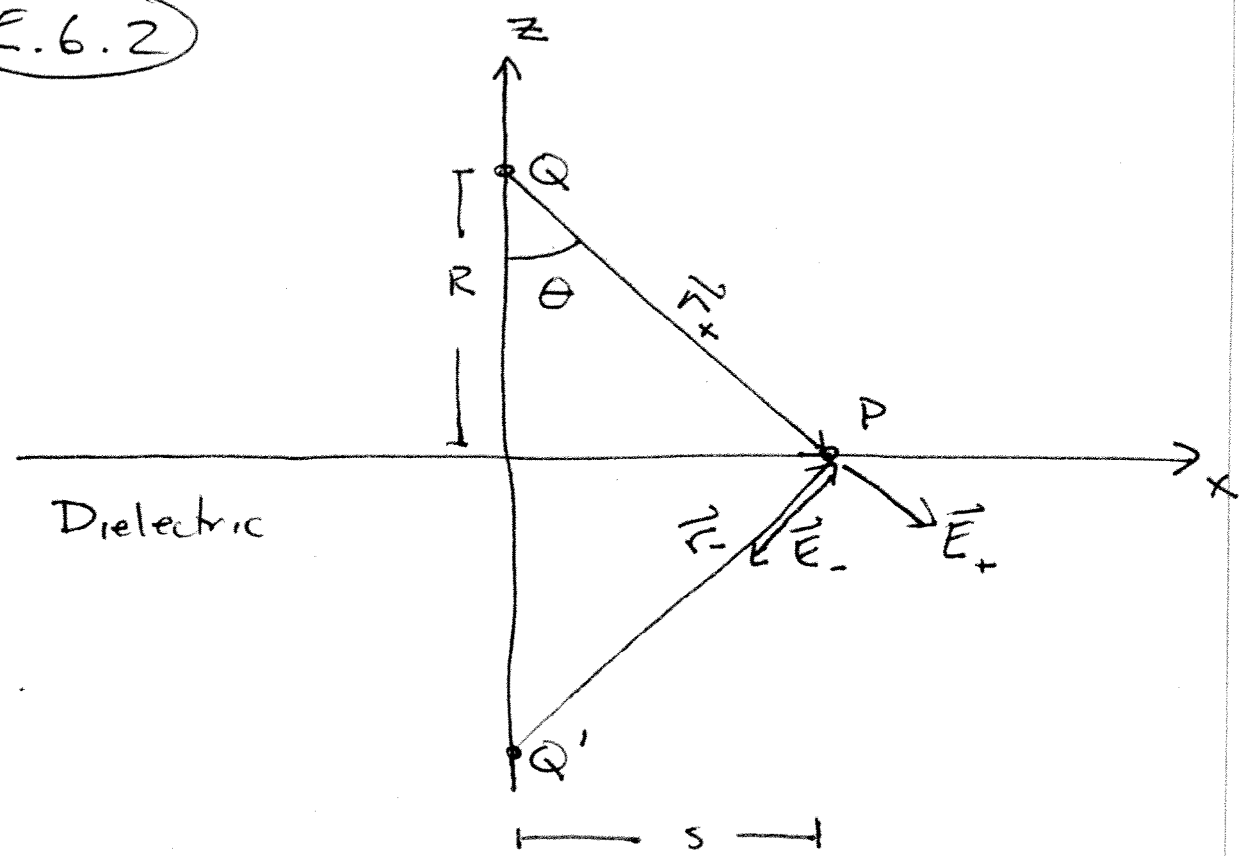
3D plot

Show contour lines



Enable interactivity

E.6.2



For $\vec{E}_t$ the field immediately above the plane, $\vec{E}_t$

$$\vec{E}_t = \vec{E}_+ + \vec{E}_- = \cancel{2|\vec{E}_+|\cos\theta}$$

$$= \frac{kQ}{r_+^3}\vec{r}_+ + \frac{kQ'}{r_-^3}\vec{r}_-$$

$$Q' = -\left(\frac{\kappa-1}{\kappa+1}\right)Q \equiv -\gamma Q$$

$$\gamma = \frac{\kappa-1}{\kappa+1}$$

$$\vec{r}_+ = s\hat{s} - R\hat{z}$$

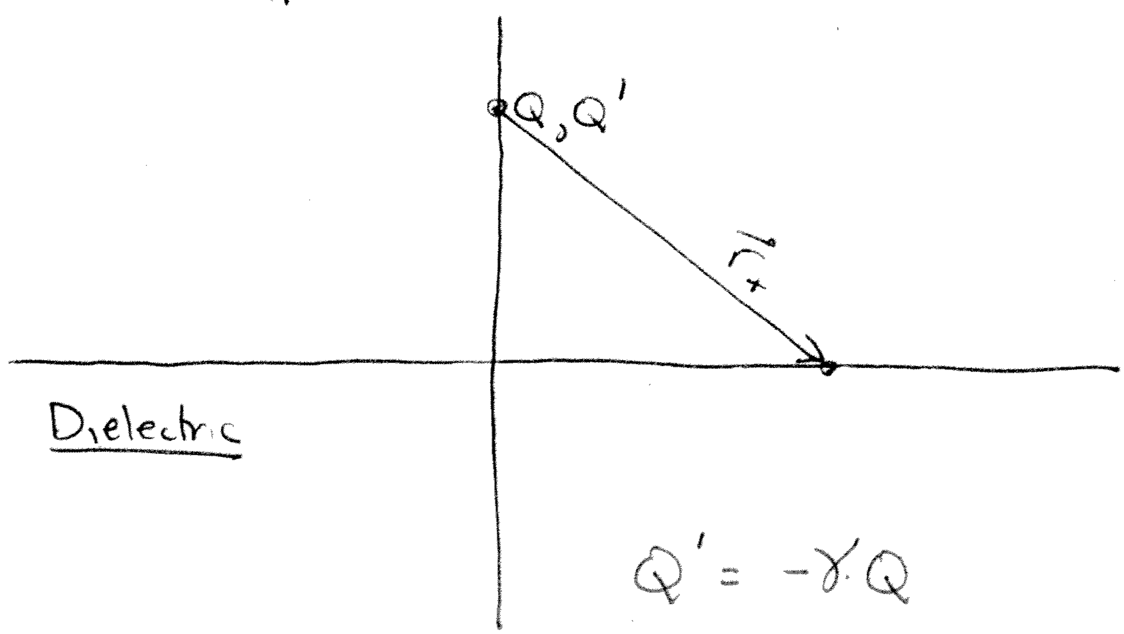
$$r_+ = \sqrt{s^2 + R^2} = r_-$$

$$\vec{r}_- = s\hat{s} + R\hat{z}$$

$$\vec{E}_t = \frac{kQ}{(s^2+R^2)^{3/2}}(s\hat{s} - R\hat{z}) - \frac{k\gamma Q}{(s^2+R^2)^{3/2}}(s\hat{s} + R\hat{z})$$

$$= \frac{kQ}{(s^2+R^2)^{3/2}}\left[s(1-\gamma)\hat{s} - (\gamma+1)R\hat{z} \right]$$

Below the plane



$$\vec{E}_b = \frac{k(Q+Q')}{r_+^3}\vec{r}_+$$

$$= \frac{kQ(1-\gamma)}{(s^2+R^2)^{3/2}} \cdot (s\hat{s} - R\hat{z})$$

<u>Charge Density</u> Use Gaussian Pillbox at surface.

$$\Phi_e = \vec{E}_t \cdot \hat{z} A - \vec{E}_b \cdot \hat{z} A = \frac{\sigma_b A}{\varepsilon_0}$$

<u>Cancel A</u>

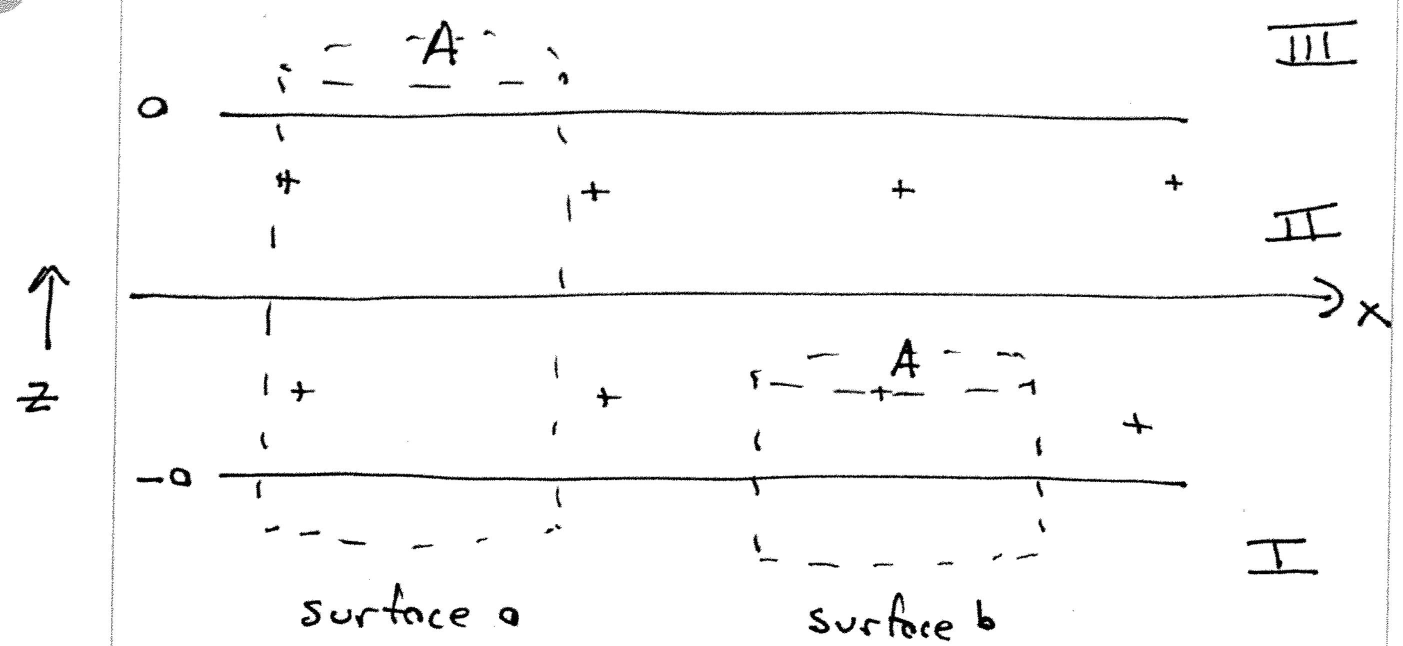
$$\vec{E}_t \cdot \hat{z} = -\frac{kQR(\gamma+1)}{(s^2+R^2)^{3/2}}$$

$$\vec{E}_b \cdot \hat{z} = -\frac{kQR(1-\gamma)}{(s^2+R^2)^{3/2}}$$

$$\sigma_b = \varepsilon_0\left(\frac{-kQR(\gamma+1)}{(s^2+R^2)^{3/2}} + \frac{kQR(1-\gamma)}{(s^2+R^2)^{3/2}}\right)$$

$$= \frac{-QR\gamma}{2\pi(s^2+R^2)^{3/2}}$$ ✓

E.6.3



Surface a   Apply Gauss' Law for Displacement

$$\Phi_d = \vec{D}_{III}\cdot\hat{z}\, A + \vec{D}_{I}\cdot(-\hat{z})\, A = Q_{fenc}$$

$$= 2aA\rho$$

$$D_{III} - D_{I} = 2a\rho$$

By symmetry $D_{III} = -D_{I}$

$$2D_{III} = 2a\rho$$

$$\vec{D}_{III} = a\rho\,\hat{z}$$

$$\vec{D}_{I} = -a\rho\,\hat{z}$$

Surface b $\qquad Q_{fenc} = (x+a) A \rho$

$$\cancel{D_{II}} \quad D_{II} - D_{I} = (x+a)\rho = \frac{Q_{fenc}}{A}$$

$$D_{II} = D_{I} + (x+a)\rho$$

$$= -a\rho + (x+a)\rho = x\rho$$

$$\vec{D}_{II} = x\rho\,\hat{z}$$

Electric Field and Polarization

In region I and III, $\vec{P} = 0$, so

$$\vec{D}_{III} = \varepsilon_0 \vec{E}_{III} \quad \text{and} \quad \vec{D}_{I} = \varepsilon_0 \vec{E}_{I}$$

$$\vec{E}_{I} = \frac{-a\rho}{\varepsilon_0}\hat{z} \qquad \vec{P}_{I} = 0$$

$$\vec{E}_{III} = \frac{a\rho}{\varepsilon_0}\hat{z} \qquad \vec{P}_{III} = 0$$

In region II, $\vec{D}_{II} = \varepsilon_0 \kappa \vec{E}_{II}$ in linear dielectric

$$\vec{E}_{II} = \frac{x\rho}{\kappa\varepsilon_0}\hat{z} \qquad \vec{P}_{II} = \varepsilon_0 \chi_e \vec{E}_{II}$$

$$= \varepsilon_0(\kappa - 1)\vec{E}_{II}$$

$$\vec{P}_{II} = \frac{x\rho}{\varepsilon_0}\,\frac{\kappa-1}{\kappa}\,\hat{z}$$

E.6.4

~~2.6~~ $\vec{P} = \gamma r^2 \hat{r}$

$$\rho_b = -\nabla \cdot \vec{P} = -\gamma \frac{1}{r^2} \frac{\partial}{\partial r} r^4$$

$$= -4\gamma r$$

Surface charge $\sigma_b = \vec{P} \cdot \hat{r} = \gamma a^2$

<u>Inside Object</u> $r < a$ Gaussian surface radius r

$$Q_{enc} = \int \rho \, d\tau$$

$$= \int \rho \, r^2 \sin\theta \, dr \, d\phi \, d\theta$$

$$= 4\pi \int_0^r \rho \, r^2 \, dr$$

$$= -16\gamma\pi \int_0^r r^3 \, dr$$

$$= -4\gamma\pi r^4$$

<u>Gauss Law</u>

$$\Phi_e = \int \vec{E} \cdot \hat{n} \, dA = 4\pi r^2 E = \frac{Q_{enc}}{\epsilon_0}$$

$$\vec{E}_i = \frac{Q_{enc}}{4\pi\epsilon_0 r^2} \hat{r} = \frac{-4\gamma\pi r^4}{4\pi\epsilon_0 r^2} \hat{r}$$

$$= -\frac{\gamma r^2}{\epsilon_0} \hat{r}$$

<u>Field Outside</u> Charge enclosed is total charge of volume charge plus surface charge.

$$Q_{enc} = Q_{vol} + Q_{surface}$$

$$= -4\gamma\pi a^4 + (\gamma a^2)(4\pi a^2)$$

$$= 0$$

$$\vec{E}_o = 0$$

An alternate, but cool, solution I didn't think of

$$Q_{f\,enc} = 0 \quad \text{everywhere}$$

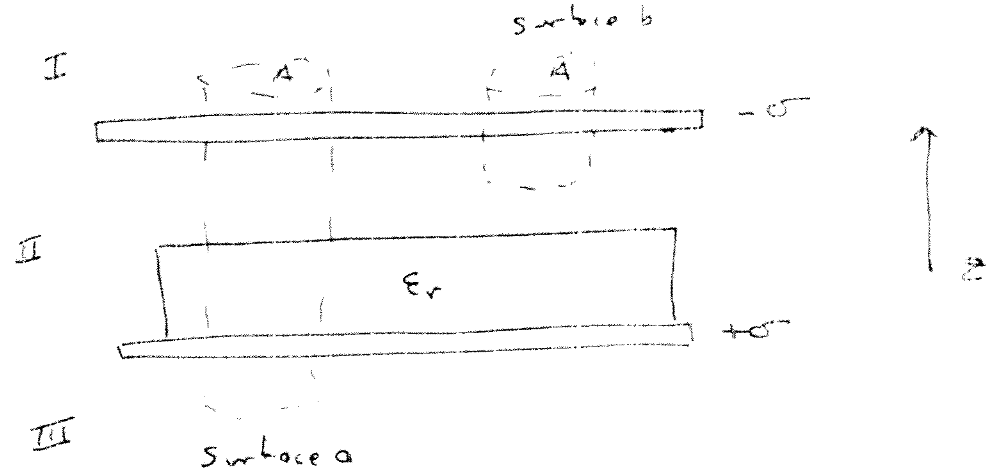
by symmetry $\Longrightarrow$ $\vec{D} = 0$ everywhere.

Outside $\quad \vec{D} = \varepsilon_0 \vec{E}_o \quad \Longrightarrow \quad \vec{E}_o = 0$

Inside $\quad \vec{D} = \varepsilon_0 \vec{E}_i + \vec{P} = 0$

$$\vec{E}_i = -\frac{\vec{P}}{\varepsilon_0} = \frac{-\gamma r^2}{\varepsilon_0}\,\hat{r}$$

E.6.5



Surface a $\quad Q_{f enc} = 0$

$$\Phi_d = D_I A - D_{III} A = 0 \qquad \text{Gauss (Displacement)}$$

By symmetry, fields equal but opposite

$$D_I = -D_{III} \quad \Rightarrow \quad \vec{D}_I = \vec{D}_{III} = 0$$

Surface b $\quad Q_{f enc} = -\sigma A$

$$D_I A - D_{II} A = Q_{f enc} = -\sigma A$$

$$\vec{D}_{II} = \sigma \hat{z} = \vec{D} \text{ inside dielectric}$$

Since dielectric linear,

$$\varepsilon_0 \varepsilon_r \vec{E}_{r} = \vec{D} = \sigma \hat{z}$$

$$\vec{E}_{r} = \frac{\sigma}{\varepsilon_0 \varepsilon_r} \hat{z}$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0(\varepsilon_r - 1)\vec{E}$$

$$= \varepsilon_0(\varepsilon_r - 1)\frac{\sigma}{\varepsilon_0 \varepsilon_r}\hat{z} = \frac{\varepsilon_r - 1}{\varepsilon_r}\sigma\hat{z}$$

Bound Charge Density

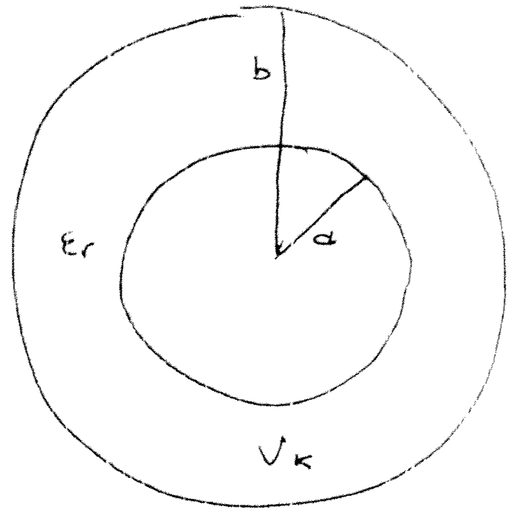
$$\rho_b = -\nabla \cdot \vec{P} = 0$$

<u>Surface Charge Densities</u>

Top surface $\sigma_t = \hat{z} \cdot \vec{P} = \frac{\varepsilon_r - 1}{\varepsilon_r}\sigma$

Bottom surface $\sigma_b = (-\hat{z}) \cdot \vec{P} = \frac{-(\varepsilon_r - 1)}{\varepsilon_r}\sigma$

E.6.6



Solution to Laplaces Eqn keep terms that don't explode.

$$V_k = \sum_n \left(A_n r^n + B_n r^{-(n+1)}\right) P_n(\cos\theta)$$

$$V_o = \sum_n C_n r^{-(n+1)} P_n(\cos\theta)$$

The applied potential is $V_0 P_1(\cos\theta)$ so only $n=1$ terms are needed.

$$V_k = \left[A_1 r + \frac{B_1}{r^2}\right] P_1(\cos\theta)$$

$$V_o = \frac{C_1}{r^2} P_1(\cos\theta)$$

Boundary Conditions

$$V_k(a, \phi) = V_0 P_1(\cos\theta)$$

$$= \left(A_1 a + \frac{B_1}{a^2}\right) P_1(\cos\theta)$$

$$V_0 = A_1 a + \frac{B_1}{a^2}$$

Continuity at b

$$V_k(b, \phi) = V_o(b, \phi)$$

$$\left(A_1 b + \frac{B_1}{b^2}\right) P_1(\cos\theta) = \frac{C_1}{b^2} P_1(\cos\theta)$$

$$A_1 b + \frac{B_1}{b^2} = \frac{C_1}{b^2}$$

Electrostatic BC

$$\varepsilon_0 \frac{\partial V_0}{\partial r}\bigg|_b - \varepsilon_r \varepsilon_0 \frac{\partial V_k}{\partial r}\bigg|_b = -\sigma_f = 0$$

$$-2\frac{C_1}{b^3} - \varepsilon_r\left(A_1 - 2\frac{B_1}{b^3}\right) = 0$$

$$V_0 a^2 = A_1 a^3 + B_1 \qquad (1)$$

$$0 = A_1 b^3 + B_1 - C_1 \qquad (2)$$

$$0 = A_1 b^3 \varepsilon_r - 2\varepsilon_r B_1 + 2C_1 \qquad (3)$$

$2(2)+(1)$

$$A_1 b^3 (2+\varepsilon_r) + (2-2\varepsilon_r) B_1 = 0$$

$$B_1 = -\frac{A_1 b^3}{2} \frac{2+\varepsilon_r}{1-\varepsilon_r} = \frac{A_1 b^3}{2} \frac{2+\varepsilon_r}{\varepsilon_r - 1} \qquad (4)$$

$(4) \rightarrow (1)$

$$V_0 a^2 = A_1 a^3 + \frac{A_1 b^3}{2} \frac{2+\varepsilon_r}{\varepsilon_r - 1}$$

$$= A_1 \left(a^3 + \frac{b^3}{2} \frac{2+\varepsilon_r}{\varepsilon_r - 1} \right)$$

$$A_1 = \frac{V_0 a^2}{a^3 + \frac{b^3}{2} \frac{2+\varepsilon_r}{\varepsilon_r - 1}} \qquad (5)$$

(5) $\rightarrow$ (4)

$$B_1 = \frac{A_1 b^3}{2} \frac{2+\varepsilon_r}{\varepsilon_r - 1}$$

$$= \frac{V_0 a^2}{a^3 + \frac{b^3}{2}\left(\frac{2+\varepsilon_r}{\varepsilon_r-1}\right)} \cdot \frac{b^3}{2} \frac{2+\varepsilon_r}{\varepsilon_r - 1}$$

$$C_1 = A_1 b^3 + B_1$$

$$= \frac{V_0 a^2 b^3}{a^3 + \frac{b^3}{2}\left(\frac{2+\varepsilon_r}{\varepsilon_r-1}\right)} + \frac{V_0 a^2 b^3}{a^3 + \frac{b^3}{2}\left(\frac{2+\varepsilon_r}{\varepsilon_r-1}\right)} \cdot \frac{2+\varepsilon_r}{2(\varepsilon_r - 1)}$$