

Homework 9

Due Wednesday 4/23/2014 - at beginning of class

Griffiths' 4 Problems (3rd Edition numbers are the same)

6.1

6.3 - Work part (b) only.

6.8

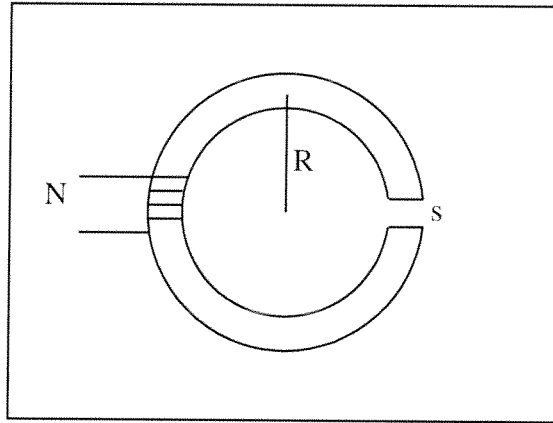
6.12

6.17

6.18

Additional Problems

E.9.1 Compute the magnetic field in the gap ($s = 0.15\text{cm}$) in a toroidal iron ring. The radius of the ring is $R = 8.1\text{cm}$. The ring is wrapped with 100 turns carrying current $I = 0.7\text{A}$. At the operating current, the relative permeability of the ring is $\mu_r = 100$.



E.9.2 Repeat problem E.9.1 where 1/10 the circumference of the iron is replaced with a permanent magnetic material with magnetization $M = 10^5\text{A/m}$. The permanent magnet produces a field in the same direction as the field produced by the wraps of wire.

E.9.3 A manufacturer of Alnico magnets reports a residual field of 12,500 Gauss. This is the field in the center of an infinitely long magnet. Compute the magnetization \vec{M} . Our cylindrical Alnico lab magnets were about 10cm long with radius 0.5cm. Compare the field at the center of the flat surface of this magnet to field at the surface of a disk magnet of height 1mm and radius 0.5cm. You may use the finite solenoid formula. Also compute the field of the disk magnet modelling the bound current as a ring of current. Modelling the two magnets as point dipoles, compare the magnetic field at 30cm in the direction of the moment of the two magnets.

E.9.4 A circular magnet with radius $a = 1\text{cm}$ and thickness $d = 1\text{mm}$ and magnetization $1 \times 10^5\text{A/m}^2$ lies in the $x - y$ plane centered at the origin. You may model the disk as infinitely thin.

a Calculate the magnetic field at the center of the magnet.

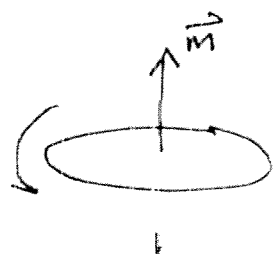
b Calculate the torque a magnetic field $\vec{B} = B_0\hat{x}$ would exert on the magnet if $B_0 = 0.2\text{T}$.

E.9.5 A cylindrical stainless steel channel with relative permeability μ_r , inner radius a and outer radius b , confines a plasma with current density

$$\vec{j} = \frac{J_0 a}{s} \sin\left(\frac{\pi s}{a}\right) \hat{z}$$

where the axis of the solenoid is the z -axis and J_0 is a constant. Compute \vec{H} , \vec{B} , and \vec{M} everywhere.

6.1



The loop will rotate until its moment points to the bottom of the page

The moment of the round loop is

$$\vec{m}_1 = \pi a^2 I \hat{y}$$

The field of loop 2 is

$$\vec{B}_{12} = -\frac{\mu_0}{4\pi} \frac{m}{x^3} \hat{y} = -\frac{\mu_0}{4\pi} \frac{\pi a^2 I}{x^3}$$

The moment of loop 2 is

$$\vec{m}_2 = b^2 I \hat{x}$$

The torque on loop 2

$$\begin{aligned} \vec{\tau}_{12} &= \vec{m}_2 \times \vec{B}_{12} \\ &= (b^2 I) \left(\frac{\mu_0}{4\pi} \frac{\pi a^2 I}{x^3} \right) (-\hat{z}) \end{aligned}$$

$$\vec{T}_{12} = - \frac{d^2 b^2 I^2}{4r^3} \hat{z}$$

6.3



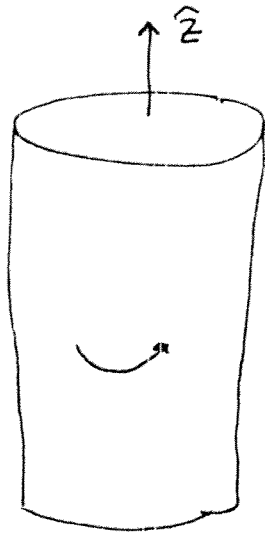
$$B_{12} = \frac{2\mu_0 m_1}{4\pi r^3}$$

$$F_{12} = \nabla (\vec{m}_2 \cdot \vec{B}_{12})$$

$$= \nabla \left(\frac{\mu_0}{2\pi} \frac{m_1 m_2}{r^3} \right)$$

$$\vec{F}_{12} = -\frac{3}{2} \frac{\mu_0}{\pi} \frac{m_1 m_2}{r^4} \hat{x}$$

6.8



$$\vec{M} = \kappa s^2 \hat{\phi}$$

The problem has azimuthal symmetry, so the magnetic field must be circular about the axis. $\Rightarrow \vec{B} = B \hat{\phi}$

There are no free currents, so

$$\oint \vec{H} \cdot d\vec{l} = 0$$

$$\Rightarrow \vec{H} = 0 \quad \text{everywhere.}$$

$$\text{Outside} \quad \vec{M}_0 = 0 \quad \mu_0 H = 0 \quad \Rightarrow \vec{B}_0 = 0$$

$$\text{Inside} \quad \mu_0 \vec{H} + \mu_0 \vec{M} = \vec{B}_i \quad \Rightarrow \quad \vec{B}_i = \mu_0 \kappa s^2 \hat{\phi}$$

(2)

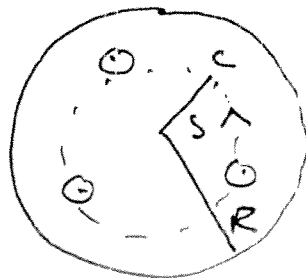
As a check, we can also attack this through the ~~no~~ bound currents.

$$\begin{aligned}
 \vec{J}_b &= \nabla \times \vec{M} \\
 &= -\frac{\partial M_\phi}{\partial z} \hat{s} + \frac{1}{s} \frac{\partial}{\partial s} s M_\phi \hat{z} \\
 &= \frac{1}{s} \frac{\partial}{\partial s} k s^3 \hat{z} \\
 &= 3k s \hat{z}
 \end{aligned}$$

Surface Current

$$\begin{aligned}
 \vec{K}_b &= \vec{M} \times \hat{s} = k s^2 \hat{\phi} \times \hat{s} \\
 &= -k s^2 \hat{z} \quad \text{since } \hat{s} \times \hat{\phi} = \hat{z}
 \end{aligned}$$

End View



Inside $I_{enc} = \int_0^s \mathbf{J}_b \cdot d\mathbf{a} \quad da = ds s d\phi$

$$= \int_0^s ds \int_0^{2\pi} d\phi s \mathbf{J}_b$$

$$= \int_0^s ds \int_0^{2\pi} d\phi 3ks^2$$

$$= 6\pi \int_0^s ds ks^2$$

$$= 2\pi ks^3$$

Field inside

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = 2\pi s B_i$$

$$B_i = \frac{\mu_0 I_{enc}}{2\pi s} = \frac{\mu_0 2\pi ks^3}{2\pi s} = \mu_0 ks^2$$

Outside $I_{enc} = 2\pi kR^3 + k_b \cdot 2\pi R$

$$= 0$$

$$\vec{B}_o = 0$$

6.12

$$\vec{M} = \kappa s \hat{z}$$

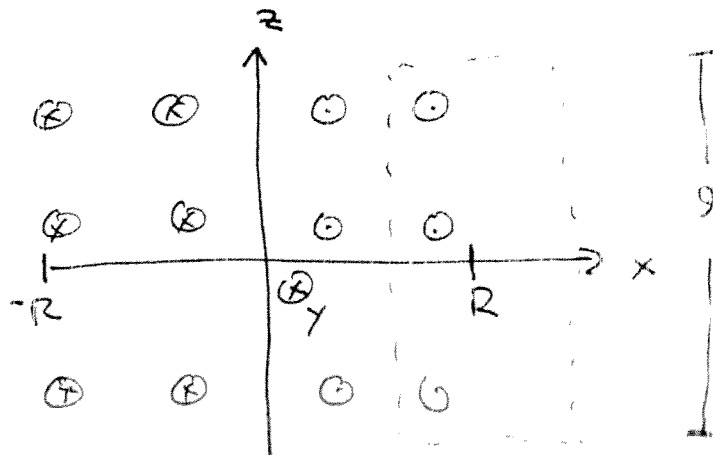
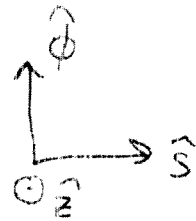
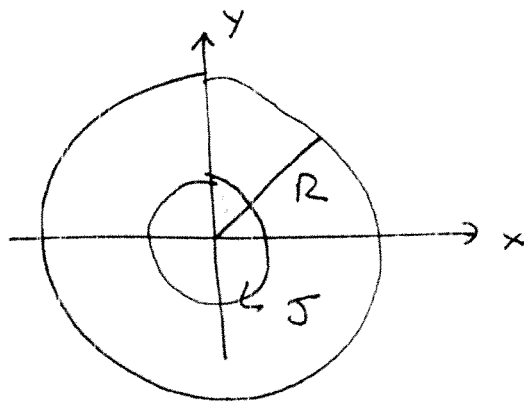
(a)

$$\vec{J}_b = \nabla \times \vec{M}$$

$$= \frac{1}{s} \frac{\partial M_z}{\partial \phi} \hat{s} - \frac{\partial M_z}{\partial s} \hat{\phi}$$

$$= -\kappa \hat{\phi}$$

$$\vec{K}_b = \vec{M} \times \hat{s} = \kappa s \hat{z} \times \hat{s} = \kappa R \hat{\phi}$$



Use an Amperian Path of length \mathcal{L} as drawn.

The system is a concentric stack of infinite solenoids so the field outside is zero using the same reasoning as the infinite solenoid.

Inside the current encircled by the Amperian path is

$$\begin{aligned} I_{\text{enc}} &= \mathcal{L} \int_s^R \mathbf{J}_b \cdot d\mathbf{s} + \mathcal{L} K b \\ &= \mathcal{L} (-K) \int_s^R ds + \mathcal{L} K R \\ &= \mathcal{L} (-K) (R - s) + \mathcal{L} K R = \mathcal{L} K s \end{aligned}$$

Ampere's Law

$$\oint \vec{B} \cdot d\vec{\mathcal{L}} = B \mathcal{L} = \mu_0 I_{\text{enc}}$$

$$\vec{B}_i = \mu_0 K s \hat{z}$$

* Note, direction is subtle, always more surface current than volume current encircled.

(b) No free currents + azimuthal symmetry

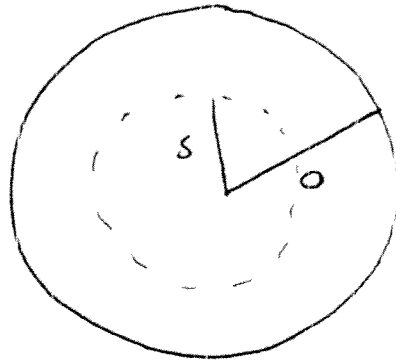
$$\oint \vec{H} \cdot d\vec{l} = 2\pi r H = 0$$

$$\vec{H} = 0 \quad \text{everywhere}$$

$$\vec{B}_0 = \mu_0 \vec{H} = 0 \quad \text{Outside}$$

$$\vec{B}_c = \mu_0 \vec{H} + \mu_0 \vec{M} = \mu_0 k s \hat{\phi}$$

6.17



Current out of
page \hat{z}

Field ccw ($+\hat{\phi}$)

Free current encircled by Amperian path of radius
 $s < a$.

$$I_{\text{enc}} = \int J \, d\mathbf{a} = \int_0^s ds \int_0^{2\pi} s \, d\phi \, J$$

$$J = \frac{I}{\pi a^2}$$

$$I_{\text{enc}} = \frac{2\pi I}{\pi a^2} \int_0^s s \, ds = \frac{2\pi I}{\pi a^2} \frac{s^2}{2}$$

$$= I \frac{s^2}{a^2}$$

Ampere's Law

$$\oint \vec{H} \cdot d\vec{l} = I_{\text{enc}} = 2\pi s H$$

Inside $s < a$

$$H_i = \frac{I_{enc}}{2\pi s} = \frac{I s^2/a^2}{2\pi s}$$

$$= \frac{I s}{2\pi a^2} = \mu_r B_i / \mu_r \mu_0$$

$$B_i = \frac{\mu_r \mu_0 I s}{2\pi a^2} = \frac{(1 + \chi_m) \mu_0 I s}{2\pi a^2}$$

Outside $s > a$ $I_{enc} = I$

$$H_o = \frac{I}{2\pi s}$$

$$B_o = \mu_0 H_o$$

$$B_o = \frac{\mu_0 I}{2\pi s}$$

Magnetization

$$\vec{M} = \chi_m \vec{H}$$

$$\vec{M}_i = \frac{\chi_m I s}{2\pi a^2} \hat{\phi}$$

Volume Bound Current

$$\vec{J}_b = \nabla \times \vec{M}$$

$$= + \frac{1}{s} \frac{\partial}{\partial s} s M_i \hat{z}$$

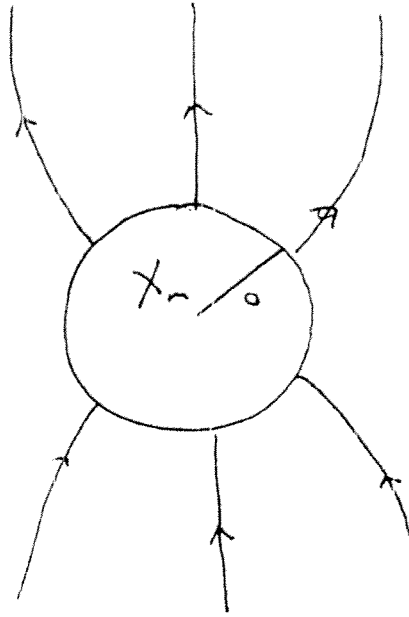
$$= + \frac{\chi_m I}{\pi a^2} \hat{z} = \chi_m \vec{J}_f$$

Surface Current

$$\vec{K}_b = \vec{M} \times \hat{s} = \frac{\chi_m I a}{2\pi a^2} \cdot (\hat{\phi} \times \hat{s})$$

$$= - \frac{\chi_m I}{2\pi a} \hat{z}$$

6.18



$$\vec{B}_\infty = B_0 \hat{z}$$

\vec{H} satisfies Laplace's eqn except at boundary.

$$\vec{H} \equiv -\nabla W$$

Boundary Conditions

$$W_i(a, \theta) = W_o(a, \theta) \quad \text{continuity}$$

$$\vec{H}_o \rightarrow \frac{\vec{B}_o}{\mu_o} = \frac{B_o}{\mu_o} \hat{z}$$

$$\rightarrow W_o \Rightarrow -\frac{B_o}{\mu_o} z \quad \text{as } z \rightarrow \infty$$

$$= -\frac{B_o}{\mu_o} r P_1(\cos\theta)$$

$$\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B}_o \cdot \hat{n} = \vec{B}_i \cdot \hat{n}$$

$$\vec{B}_o = \mu_o \vec{H}_o \quad \vec{B}_i = \mu_o \mu_r \vec{H}_i$$

$$\vec{H}_o \cdot \hat{n} = \mu_r \vec{H}_i \cdot \hat{n}$$

$$\left. \frac{\partial W_o}{\partial r} \right|_a = \mu_r \left. \frac{\partial W_i}{\partial r} \right|_a$$

Magnetic Potential

$$W_i = \sum A_n r^n P_n(\cos \theta) \quad \text{inside}$$

$$W_o = -\frac{B_o}{\mu_o} r P_1(\cos \theta) + \sum C_n r^{-(n+1)} P_n(\cos \theta)$$

Clearly, only $n=1$ term will survive,

$$W_i = A_1 r P_1(\cos \theta)$$

$$W_o = -\frac{B_o}{\mu_o} r P_1(\cos \theta) + \frac{C_1}{r^2} P_1(\cos \theta)$$

Continuity

$$W_i(a, \theta) = W_o(a, \theta)$$

$$A_1 a = -\frac{B_o}{\mu_o} a + \frac{C_1}{a^2}$$

ES BC

$$\left. \frac{\partial W_o}{\partial r} \right|_a = -\frac{B_o}{\mu_o} P_1 - \frac{2C_1}{a^3} P_1$$

$$= \mu_r A_1 P_1 = \mu_r \left. \frac{\partial W_i}{\partial r} \right|_a$$

$$C_1 = A_1 a^3 + \frac{B_o}{\mu_o} a^3$$

$$2C_1 = -\frac{B_o}{\mu_o} a^3 - \mu_r A_1 a^3$$

$$3C_1 = A_1 (1 - \mu_r) a^3$$

$$A_1 =$$

$$2C_1 + \mu_r C_1 = -\frac{B_0}{\mu_0} a^3 + \mu_r \frac{B_0}{\mu_0} a^3$$

$$C_1 = \frac{B_0}{\mu_0} a^3 \left(\frac{\mu_r - 1}{\mu_r + 2} \right)$$

$$A_1 = \frac{3C_1}{(1 - \mu_r)a^3} = -\frac{B_0}{\mu_0} \frac{3}{\mu_r + 2}$$

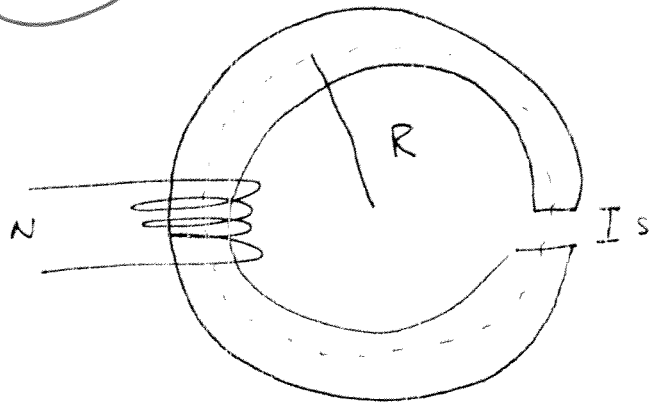
Potentials

$$\frac{\text{Inside } P}{W_i} = A_1 r \cos \theta = -\frac{B_0}{\mu_0} \frac{3}{\mu_r + 2} z$$

$$\vec{H}_i = -\nabla W_i = \frac{B_0}{\mu_0} \frac{3}{\mu_r + 2} \hat{z}$$

$$\vec{B}_i = \mu_r \mu_0 \vec{H}_i = \mu_r B_0 \frac{3}{\mu_r + 2} \hat{z}$$

~~E.1~~ E.9.1



$$\oint \vec{H} \cdot d\vec{l} = NI$$
$$= (2\pi R - s) H_i + s H_o$$

Since $\nabla \cdot \vec{B} = 0$ $B_o = B_i = B$

$$H_o = B/\mu_0 \quad B_i = \mu_0 H_i + \mu_0 M$$

$$H_i = B/\mu_0 \mu_r$$

$$NI = (2\pi R - s) \frac{B}{\mu_0 \mu_r} + s \frac{B}{\mu_0}$$

$$\mu_0 \mu_r NI = (2\pi R - s) B + B \mu_r s$$

$$= B(2\pi R - s + \mu_r s)$$

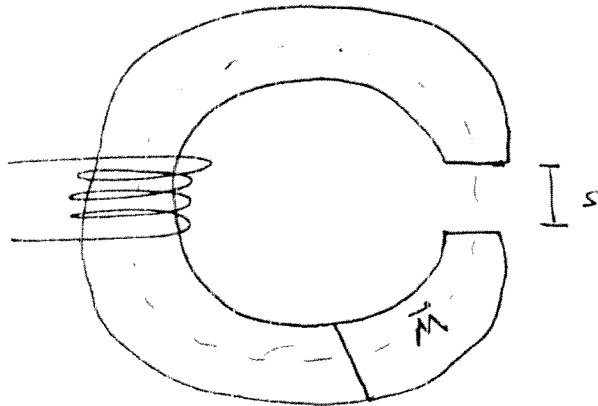
$$B = \frac{\mu_0 \mu_r N I}{2\pi R - s + \mu_r s}$$

$$= \frac{(4\pi \times 10^{-7} \frac{Tm}{A}) (100) (100) (0.7A)}{2\pi (0.081m) - 0.0015m + (100)(0.0015m)}$$

$$= 0.013 T$$

~~E2~~

E.9.2



H_0 = Field in Gap

H_i = Field in Iron

H_m = Field in magnetic

$\nabla \cdot \vec{B} = 0 \Rightarrow$ Magnetic field in all are equal.

$$H_0 = B / \mu_0 \quad H_i = \frac{B}{\mu_r \mu_0}$$

$$B = \mu_0 H_m + \mu_0 M \Rightarrow H_m = \frac{B - \mu_0 M}{\mu_0}$$

Recall, H_m points in opposite direction to B .

$$\begin{aligned} \text{mmf} = \oint \vec{H} \cdot d\vec{l} &= NI = s H_0 + \frac{1}{\mu_0} \cdot (2\pi R - s) H_m \\ &+ \frac{\mu_0}{\mu_0} (2\pi R - s) H_i = NI \end{aligned}$$

$$s \frac{B}{\mu_0} + \frac{1}{10} (2\pi R - s) \left(\frac{B - \mu_0 M}{\mu_0} \right) + \frac{9}{10} (2\pi R - s) \frac{B}{\mu_0 \mu_r} = NI$$

$$10s \mu_r B + \mu_r (2\pi R - s) (B - \mu_0 M) + 9(2\pi R - s) B = \mu_r \mu_0 NI \cdot 10$$

$$10s \mu_r B + \mu_r (2\pi R - s) B + 9(2\pi R - s) B = 10\mu_r \mu_0 NI + \mu_0 \mu_r M (2\pi R - s)$$

$$B (10s \mu_r + (\mu_r + 9)(2\pi R - s)) =$$

$$B = \frac{\mu_0 \mu_r (10NI + M(2\pi R - s))}{10s \mu_r + (\mu_r + 9)(2\pi R - s)}$$

$$= \frac{\mu_0 \cdot 100 \left(10 \cdot 100 \cdot 0.7A + 10^5 \frac{A}{m} \cdot (2\pi (0.081m) - 0.0015m) \right)}{10 \cdot 0.0015m \cdot 100 + (109) (2\pi (0.081m) - 0.0015m)}$$

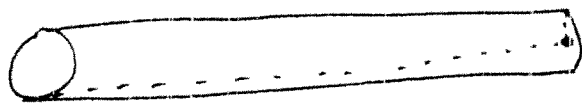
$$= \frac{\mu_0 \cdot 100 \cdot (700A + 50743A)}{1.5m + 55.3m} = 0.14T$$

~~E3~~ E.9.3
 $B_r = 1.25 \text{ T}$

Infinitely long magnet, with no free currents

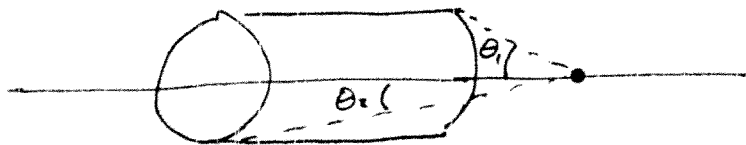
$$\vec{H} = 0 \quad \vec{B} = \mu_0 \vec{M}$$

$$M = \frac{B_r}{\mu_0} = 9.9 \times 10^5 \text{ A/m}$$



Surface current $K_b = |\vec{M} \times \hat{n}| = 9.9 \times 10^5 \text{ A/m}$

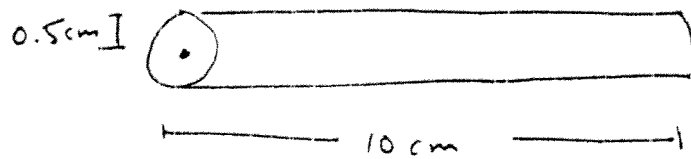
Finite Solenoid Formula



$$B = \frac{\mu_0 K_b}{2} (\cos \theta_2 - \cos \theta_1)$$

$$B = 0.625 \text{ T} (\cos \theta_2 - \cos \theta_1)$$

Long Magnet



$$\theta_1 = \frac{\pi}{2} \quad \theta_2 = \tan^{-1}\left(\frac{0.5 \text{ cm}}{10 \text{ cm}}\right) = 2.86^\circ$$

$$B = (0.625 \text{ T})(\cos 2.86^\circ - \cos 90^\circ) \\ = 0.624 \text{ T}$$

Disk Magnet



$$\theta_1 = \pi/2$$

$$\theta_2 = \tan^{-1}\left(\frac{0.5 \text{ cm}}{0.1 \text{ cm}}\right)$$

$$= 78.7^\circ$$

$$B = (0.625 \text{ T})(\cos 78.7^\circ) \\ = 0.123 \text{ T}$$

Long Magnet Moment

$$m = MV = M \pi r^2 h$$

$$= (9.9 \times 10^5 \text{ A/m}) \pi (0.005 \text{ m})^2 (0.1 \text{ m})$$

$$= 7.78 \text{ A m}^2$$

Field

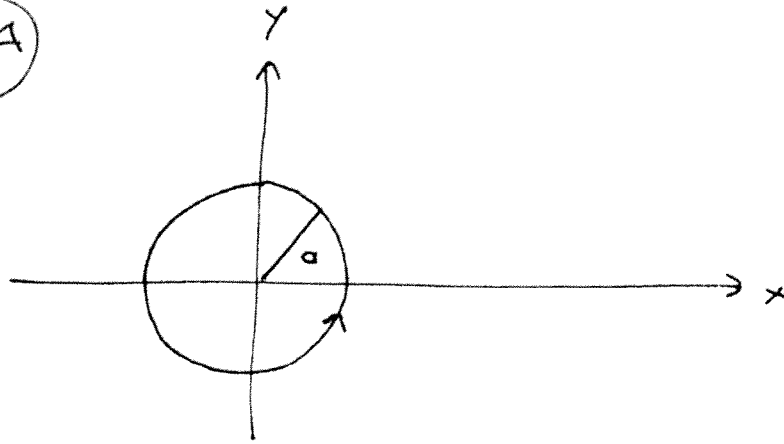
$$B = \frac{\mu_0}{4\pi} \frac{m}{r^3} = \frac{(1 \times 10^{-7} \frac{\text{Tm}}{\text{A}}) (7.78 \text{ A m}^2)}{(0.3 \text{ m})^3}$$
$$= 2.9 \times 10^{-5} \text{ T}$$

Disk Magnet

$$m = M \pi r^2 h = 0.0778 \text{ A m}^2$$

$$B = 2.9 \times 10^{-7} \text{ T}$$

9.4



The bound current density $\vec{K}_b = \vec{M} \times \hat{n} = M_0 \hat{\phi}$

The total bound current is $I = K_b d = M_0 d = 100 \text{ A}$.

The field at the center is found by integrating the current around the loop

$$\vec{B} = \oint_C d\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \hat{r}}{a^2}$$

$$= \frac{\mu_0 I}{4\pi a^2} \hat{z} \int_0^{2\pi} a d\phi$$

$$d\vec{l} = a d\phi \hat{\phi}$$

$$\hat{r} = -\hat{r}$$

$$\hat{\phi} \times -\hat{r} = \hat{z}$$

$$= \frac{\mu_0 I}{4\pi a} \cdot 2\pi \hat{z} = \frac{\mu_0 I}{2a} \hat{z} = \frac{\mu_0 M_0 d}{2a} \hat{z}$$

$$= \frac{\left(4\pi \times 10^{-7} \frac{\text{Tm}}{\text{A}}\right) (100 \text{ A})}{2(0.01 \text{ m})} \hat{z} = 2\pi \times 10^{-3} \text{ T} \hat{z}$$

(b) The moment is $\vec{m} = M_0 V \hat{z} = M_0 \pi a^2 l \hat{z}$

where I used the right-hand rule for the moment to get the direction.

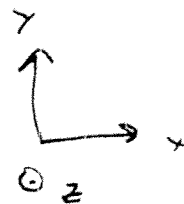
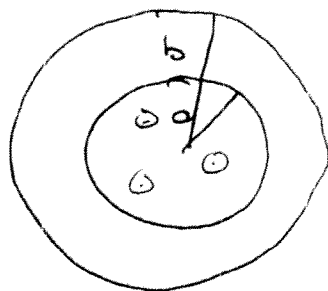
$$\vec{\tau} = \vec{m} \times \vec{B} = B_0 M_0 \pi a^2 l \hat{z} \times \hat{x}$$

$$= B_0 M_0 \pi a^2 l \hat{y}$$

$$= \left(\frac{1}{5} T\right) \left(10^5 \frac{A}{m}\right) (\pi) (0.01m)^2 (0.001m) \hat{y}$$

$$= \frac{\pi}{5} \times 10^{-2} \hat{y} = 0.00628 \text{ Nm} \hat{y}$$

~~E.9.5~~ E.9.5



Compute \vec{H}

$$\begin{aligned} \underline{s < a} \quad I_{\text{enc}} &= \int_0^{2\pi} d\phi \int_0^s J ds \\ &= 2\pi \cdot J_0 a \cdot \int_0^s \sin \frac{\pi s}{a} ds \\ &= -2\pi J_0 a \cdot \frac{a}{\pi} \cos \frac{\pi s}{a} \Big|_0^s \\ &= 2 J_0 a^2 \left(1 - \cos \frac{\pi s}{a} \right) \end{aligned}$$

Ampere's Law

$$\int \vec{H} \cdot d\vec{l} = I_{\text{enc}} = 2\pi s H$$

$$\vec{H} = \frac{I_{\text{enc}}}{2\pi s} \hat{\phi} \stackrel{\text{RHA}}{=} \frac{2 J_0 a^2}{2\pi s} \left(1 - \cos \frac{\pi s}{a} \right) \hat{\phi}$$

Since there is no material, $\vec{M} = 0$ $s < a$, and

$$\vec{B} = \mu_0 \vec{H} = \frac{J_0 a^2 \mu_0}{\pi s} \left(1 - \cos \frac{\pi s}{a} \right)$$

For $s > a$

$$\begin{aligned} I_{\text{enc}} &= 2 J_0 a^2 \left(1 - \cos \frac{\pi a}{s} \right) \\ &= 4 J_0 a^2 \end{aligned}$$

$$\vec{H} = \frac{I_{\text{enc}}}{2\pi s} \hat{\phi} = \frac{J_0 a^2}{\pi s} \hat{\phi}$$

For $s > b$, $\vec{M} = 0$, $\vec{B} = \mu_0 \vec{H} = \frac{J_0 a^2 \mu_0}{\pi s} \hat{\phi}$

For $0 < s < b$, $\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H}$

$$\vec{B} = \frac{\mu_r \mu_0 J_0 a^2}{\pi s} \hat{\phi}$$

$$\vec{M} = \chi_m \vec{H} = (\mu_r - 1) \vec{H}$$

$$= \frac{2(\mu_r - 1) J_0 a^2}{\pi s} \hat{\phi}$$