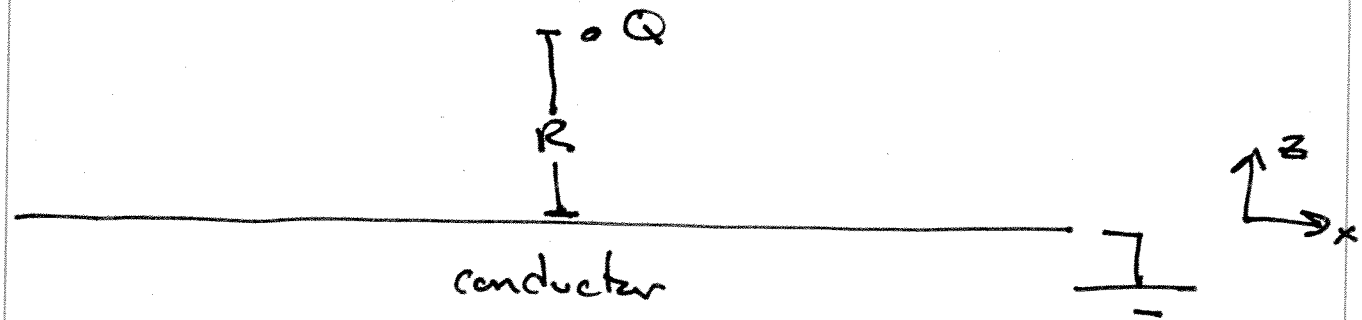
Method of Images

The method of images solves Poisson's Eqn in a region R by satisfying the boundary conditions by placing a fictitious charge outside of the region R.

Ex Consider a point charge a distance R above a grounded conducting plane.



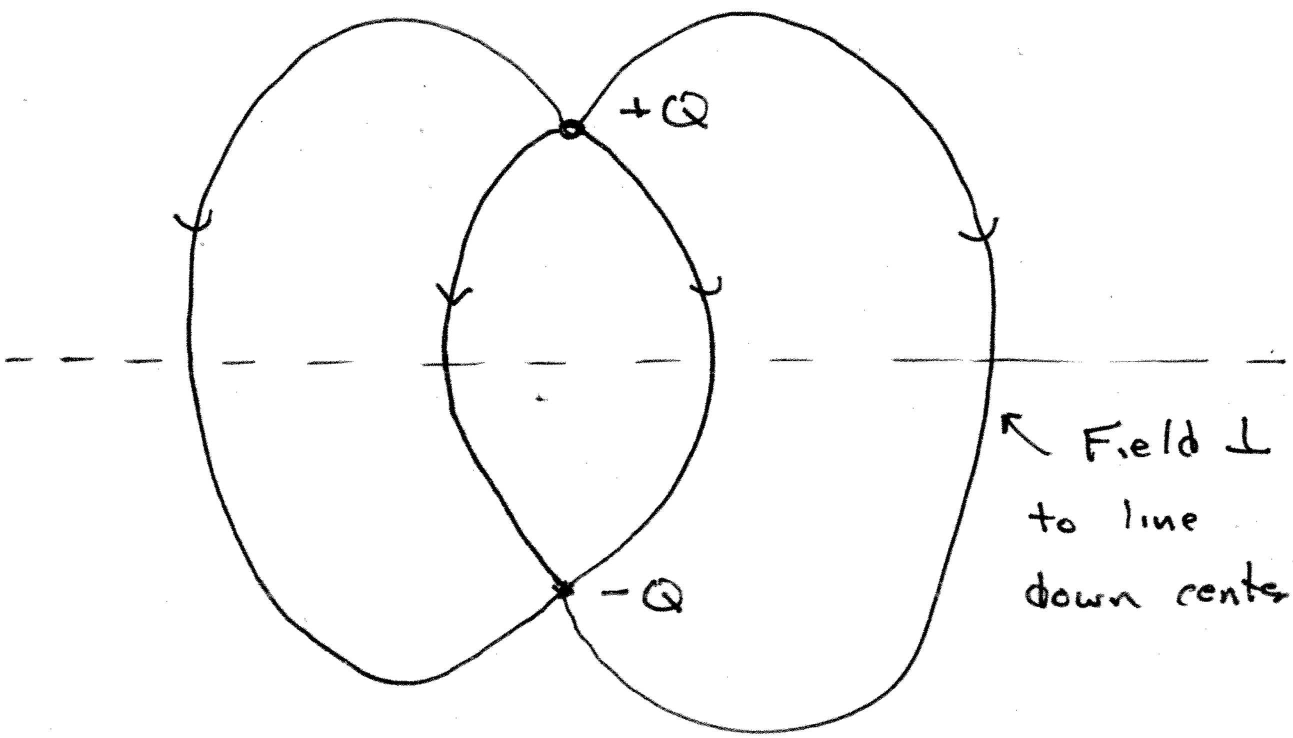
The boundary conditions for the region $z>0$ are

$$V(x,y,0) = 0$$

$$V(x,y,z) \to 0 \quad \text{as} \quad z \to \infty$$

The uniqueness thm guarantees that if we can find a solution to Poisson's Eqn that satisfies the boundary conditions that it is the unique solution.

<u>Consider the field of a dipole</u>



The potential of the two charges is

$$V = \frac{kQ}{r_+} + \frac{-kQ}{r_-}$$

All points on the plane are equidistant from both charges and therefore have $V=0$. This satifies the boundary condition for the single point charge above the conducting plane.

By the Uniqueness Thm, the potential of a point charge $+Q$ a distance R above a grounded conducting plane is

$$V = \frac{KQ}{r_+} + \frac{-KQ}{r_-}$$

The charge $-Q$ is called an <u>image charge</u>.

The image charge must be placed <u>outside</u> the region where we need the potential.

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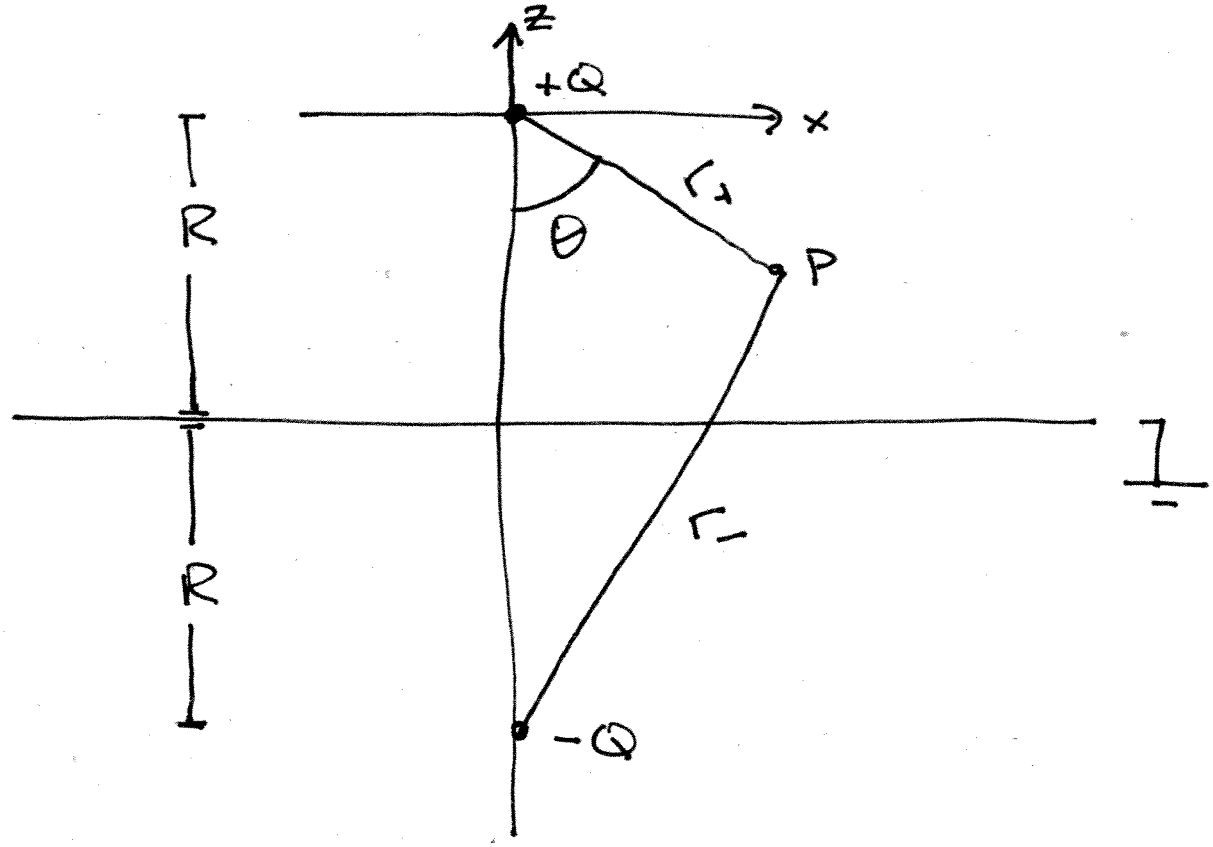
Lets work out r_+, r_-. Set the origin at $+Q$. Let θ be the angle with the z-axis.



$$V(r_+, \theta) = \frac{kQ}{r_+} - \frac{kQ}{r_-}$$

<u>Law of Cosines</u>

$$r_-^2 = r_+^2 + 4R^2 - 4r_+R\cos\theta$$

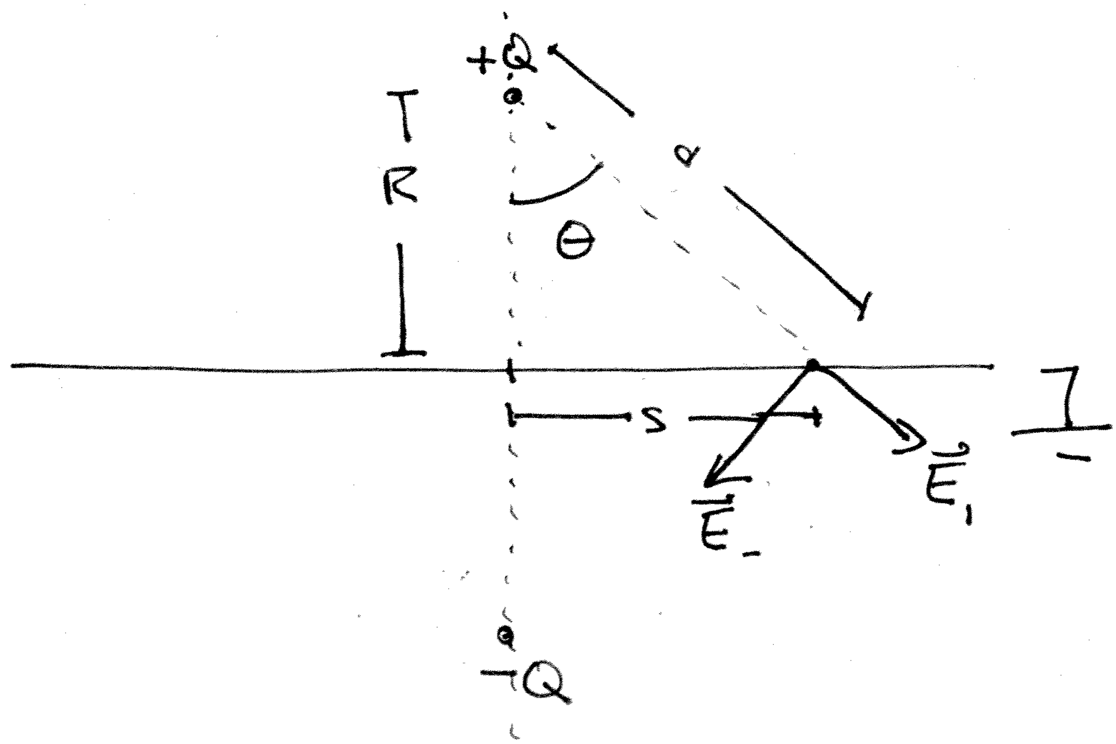
$$= r_+^2 \left(1 + \left(\frac{2R}{r_+}\right)^2 - \frac{4R}{r_+}\cos\theta\right)$$

$$V(r_+, \theta) = \frac{kQ}{r_+}\left(1 - \frac{1}{\sqrt{1 + \left(\frac{2R}{r_+}\right)^2 - \frac{4R}{r_+}\cos\theta}}\right)$$

Since the potential of the charge above the grounded plane is the same a two points charges, the fields are the same and therefore the forces.

Compute Field $\quad \vec{E} = \vec{E}_+ + \vec{E}_-$

At the conducting plane,



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At Plane, $\vec{E}_{plane} = \vec{E}_+ + \vec{E}_-$

$$\vec{E}_{plane} = -2|E_+|\cos\theta\ \hat{z}$$

$$E_+ = \frac{kQ}{d^2} = \frac{kQ}{s^2+R^2}$$

$$d = \sqrt{s^2+R^2}$$

$$\cos\theta = \frac{R}{d} = \frac{R}{\sqrt{s^2+R^2}}$$

$$\vec{E}_{plane} = -\frac{2kQR}{(s^2+R^2)^{3/2}}\ \hat{z}$$

$$= -\frac{QR}{2\pi\varepsilon_0(s^2+R^2)^{3/2}}\ \hat{z}$$

<u>Charge Density on Plane (Gaussian Pillbox)</u>

$$\phi = \vec{E}_{plane}\cdot\hat{z}\,A = \frac{\sigma A}{\varepsilon_0}$$

$$-\frac{QR}{2\pi\varepsilon_0(s^2+R^2)^{3/2}} = \frac{\sigma}{\varepsilon_0}$$

$$\sigma = \frac{-QR}{2\pi(s^2+R^2)^{3/2}}$$

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The total charge on the plane must be $-Q$ since all field lines ending on the plane end on the image charge

Check

$$Q_{plane} = \int_{plane} \sigma \, da$$

$$= \int_0^{2\pi} d\phi \int_0^{\infty} s\,ds\,\sigma$$

$$= 2\pi \int_0^{\infty} s\,ds\,\sigma$$

$$= -QR \int_0^{\infty} \frac{s\,ds}{(s^2+R^2)^{3/2}}$$

$$u = s^2+R^2 \qquad du = 2s\,ds$$

$$Q_{plane} = -\frac{QR}{2} \int_{R^2}^{\infty} \frac{du}{u^{3/2}}$$

$$= -\frac{QR}{2} \left(\frac{u^{-1/2}}{-1/2} \right)_{R^2}^{\infty}$$

$$Q_{plane} = -\frac{QR}{\sqrt{R^2}} = -Q \quad \checkmark$$

The total energy of the $+Q$ / plane system is

$$U = \frac{1}{2}\left(\frac{kQ^2}{2R}\right)$$

which is ½ the energy of the $\pm$ dipole because ½ the fields do not actually exist

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Force exerted by plane on charge

$$\vec{F} = -\frac{kQ^2}{(2R)^2}\hat{z}$$

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