Introduction to Electromagnetic Waves

Maxwell's Eqns in Vacuum ($\rho = 0$, $\vec{J} = 0$)

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Try to separate these equations. This will produce a set of equations that are not as general as Maxwell's eqns.

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \qquad \text{(identity)}$$

$$= -\nabla \times \frac{\partial \vec{B}}{\partial t}$$

$$= -\frac{\partial}{\partial t} \nabla \times \vec{B} = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

Since $\nabla \cdot \vec{E} = 0$,

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Likewise,

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = -\nabla^2 \vec{B}$$

$$= \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \times \vec{E}$$

$$= -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t} = 0$$

$$\nabla^2 \vec{B} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t} = 0$$

Or in component form (if E_x has only x dependance)

$$\frac{\partial^2 E_x}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0$$

$\Longrightarrow$ This is an example of a wave equation

More generally,

$$\nabla^2 E_x - \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0 \quad \text{3D Wave Eqn}$$

Wave Equation

The wave equation has the form

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

and has solution

$$f = g(x - vt) \qquad +x \text{ traveling}$$

$$f = h(x + vt) \qquad -x \text{ traveling}$$

The wave profile maintains the shape given by g, h as the wave travels with velocity v.

Speed of Electromagnetic Wave

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \equiv c$$

Speed of Light (c)

$$c = 299{,}792{,}458 \text{ m/s} \quad \text{exactly}$$

$$\sim 3.0 \times 10^8 \text{ m/s}$$

The most useful solutions of the wave equation are sines and cosines.

$$f(x,t) = A\cos(kx - \omega t + \delta) \quad +x \text{ traveling}$$

$$= A\cos(kx + \omega t + \delta) \quad -x \text{ traveling}$$

Amplitude (A) - Height of Wave

Phase, Phase Constant, Phase Shift (δ) - The wave is delayed a distance δ/k from reaching the origin at $t=0$.

Wavelength (λ) - Distance for one oscillation

Wavenumber (k) - $\quad k = \frac{2\pi}{\lambda}$

Period (T) - Time for one oscillation

Frequency (f) - Oscillations per second

$$f = \frac{1}{T}$$

Angular Frequency (ω) - $\omega = 2\pi f$

Complex Numbers

Complex numbers make working with EM waves profoundly easier.

$$i^2 = -1$$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$z = a + ib$$

$$\mathrm{Re}(z) = a \qquad \mathrm{Im}(z) = b$$

Since the real and imaginary parts of a complex number are independent, we can solve an equation in terms of a complex function $\tilde{f}(z)$

$$\frac{\partial^2 \tilde{f}}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \tilde{f}}{\partial t^2} = 0$$

and the physical solution will be the real part

$$f(x) = \mathrm{Re}(\tilde{f}(z))$$

Complex Waves

$$\hat{f}(x,t) = \hat{A}\, e^{i(kx-\omega t)}$$

$$\hat{A} = A e^{i\delta}$$

$$\mathrm{Re}(\hat{f}) = A \cos(kx-\omega t+\delta)$$

Wave Speed - A wave is formed of an envelope g(x-vt) which maintains it shape as the wave moves. If we look at a single point in that envelope $g(x_0)$, the location of x_0 is given by $x_0 = x - vt = kx - \omega t + \delta$

Taking the differential

$$0 = dx - v\,dt \quad \text{or} \quad 0 = k\,dx - \omega\,dt$$

$$\frac{dx}{dt} = v = \frac{\omega}{k}$$

Wave Speed $v = \frac{\omega}{k}$ (Phase Velocity)

Consider a complex solution to Maxwell's solution representing a wave propogating in the $+z$ direction.

$$\vec{E} = \vec{E}_0\, e^{i(kz-\omega t)}$$

$$\vec{B} = \vec{B}_0\, e^{i(kz-\omega t)}$$

$$\vec{E}_0,\ \vec{B}_0 \text{ complex}$$

<u>Try in Maxwell's Eqns</u>

<u>Gauss</u>

$$\nabla\cdot\vec{E} = 0 \Rightarrow ikE_{oz}e^{i(kz-\omega t)} = 0$$

$$\Rightarrow E_{oz} = 0$$

<u>No Mag Monopoles</u>

$$\nabla\cdot\vec{B} = 0 = ikB_{oz}\, e^{i(kz-\omega t)} = 0$$

$$\Rightarrow B_{oz} = 0$$

$\Rightarrow \vec{E}_0, \vec{B}_0\ \perp$ to direction of propogation

Faraday's Law $\quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{ox} e^{i(kz-\omega t)} & E_{oy} e^{i(kz-\omega t)} & 0 \end{vmatrix}$$

$$= \hat{x}\left(-ik E_{oy} e^{i(kz-\omega t)}\right)$$

$$-\hat{y}\left(-ik E_{ox} e^{i(kz-\omega t)}\right)$$

$$= -\frac{\partial \vec{B}}{\partial t} = -\left(-i\omega \hat{x} B_{ox} e^{i(kz-\omega t)}\right.$$

$$\left. - i\omega B_{oy} \hat{y} e^{i(kz-\omega t)}\right)$$

$$\Longrightarrow \quad -k E_{oy} = \omega B_{ox}$$

$$k E_{ox} = \omega B_{oy}$$

This condition can be written as

$$\vec{B}_0 = \frac{k}{\omega}\left(\hat{z} \times \vec{E}_0\right)$$

$$= \frac{1}{c}\left(\hat{z} \times \vec{E}_0\right)$$

$$\Rightarrow \quad |\vec{B}_0| = \frac{|\vec{E}_0|}{c}$$

$$\Rightarrow \quad \vec{B}_0 \perp \vec{E}_0$$

Now let the wave propagation in arbitrary direction $\hat{k}$.

Dfn Wave Vector $(\vec{k})$

$$\vec{k} = k\hat{k} = (k_x, k_y, k_z)$$

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \vec{r} = (x, y, z)$$

Dfn Polarization Direction ($\hat{n}$) - the direction of the electric field.

$$\vec{E} = E_0 \hat{n} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

The direction of $\vec{B}$ is $\hat{k} \times \hat{n}$

and $c|\vec{B}_0| = |\vec{E}_0|$

so

$$\vec{B} = \frac{E_0}{c} e^{i(\vec{k}\cdot\vec{r} - \omega t)} (\hat{k} \times \hat{n})$$

$$= \frac{1}{c} \hat{k} \times \vec{E}$$