Spherical Coordinates Using Vectors

Ex A potential $V(r,\theta) = V_0 \cos\theta + V_1 \cos 2\theta$ is established on the surface of a sphere of radius a. Compute the potential inside the sphere.

Sln Since we are at $r<a$, discard $r^{-(n+1)}$ terms because they blow up at the origin

General Solution

$$V(r,\theta) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta)\, r^n$$

B.C.

$$V(a,\theta) = V_0 \cos\theta + V_1 \cos 2\theta$$

$$= \sum A_n a^n P_n(\cos\theta)$$

Brute Force Multiple by $P_m(\cos\theta)$ and integrate (Fourier's Trick)

$$\frac{2 A_m a^m}{2m+1} = \int_{-1}^{1} P_m(\cos\theta)\left(V_0 \cos\theta + V_1 \cos 2\theta\right) d\cos\theta$$

It is worth a moment to try to exploit the vector character of $P_n(\cos\theta)$.

Try to write $\cos\theta$, $\cos 2\theta$ in terms of $P_n(\cos\theta)$.

<u>Look up P_n</u>

$$P_0 = 1 \qquad P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

<u>Work on $\cos 2\theta$</u>

$$\cos^2\theta = \frac{2P_2(\cos\theta)+1}{3}$$

$$\frac{\cos 2\theta + 1}{2} = \cos^2\theta \qquad \text{Double angle formula}$$

$$\frac{\cos 2\theta + 1}{2} = \frac{2P_2(\cos\theta)+1}{3}$$

$$\cos 2\theta = \frac{4}{3}P_2(\cos\theta) + \frac{2}{3} - 1$$

$$= \frac{4}{3}P_2(\cos\theta) - \frac{1}{3}$$

$$\frac{1}{3} = \frac{1}{3} P_0(\cos\theta)$$

$$\cos 2\theta = \frac{4}{3} P_2(\cos\theta) - \frac{1}{3} P_0(\cos\theta)$$

$$V(a,\theta) = -\frac{V_1}{3} P_0(\cos\theta) + V_0 P_1(\cos\theta)$$

$$+ \frac{4}{3} V_1 P_2(\cos\theta)$$

$$= \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta)$$

$\Longrightarrow$ The $P_n(\cos\theta)$ act like unit vectors, so

$$-\frac{V_1}{3} = A_0 a^0 \quad \Longrightarrow \quad A_0 = -\frac{V_1}{3}$$

$$V_0 = A_1 a \quad \Longrightarrow \quad A_1 = \frac{V_0}{a}$$

$$\frac{4}{3} V_1 = A_2 a^2 \quad \Longrightarrow \quad A_2 = \frac{4V_1}{3a^2}$$

<u>General Solution</u>

$$V(r,\theta) = -\frac{V_1}{3} + \frac{V_0}{a} r P_1(\cos\theta)$$

$$+ \frac{4}{3} V_1 r^2 P_2(\cos\theta)$$

$$V(r,\theta) = \frac{-V_1}{3} + \frac{V_0}{a} r \cos\theta$$

$$+ \frac{4}{3a^2} V_1 \, r^2 \cdot \frac{1}{2}\left(3\cos^2\theta - 1\right)$$

$$= -\frac{V_1}{3} - \frac{2}{3} V_1 \frac{r^2}{a^2} + \frac{V_0}{a} r \cos\theta$$

$$+ \frac{2}{3} V_1 \frac{r^2}{a^2} \cos^2\theta$$

$$= \frac{-V_1}{3} - \frac{2}{3} V_1 \frac{r^2}{a^2} + \frac{V_0 z}{a}$$

$$+ \frac{2}{3} V_1 \frac{z^2}{a^2}$$