Laplace's Eqn - Spherical Symmetry

$$\nabla^2 V = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

Radial Symmetry - B.C. independent of ϕ, θ

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

Let $V = r^n$, $\frac{\partial V}{\partial r} = n r^{n-1}$

$$\frac{\partial}{\partial r}\left(r^2 n r^{n-1}\right) = 0$$

$$\frac{\partial}{\partial r} r^{n+1} = 0$$

$\Longrightarrow$ $n = -1$

<u>Solutions</u> $1, \; 1/r$

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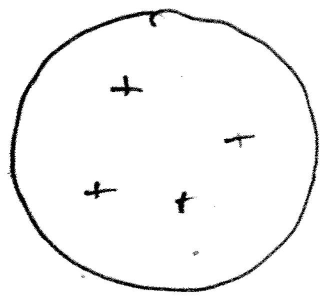
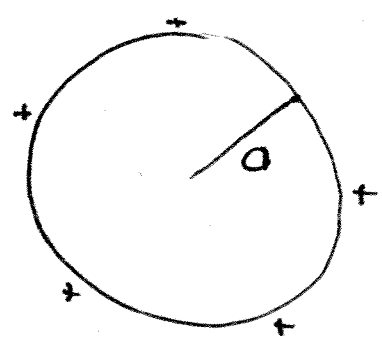
Ex Spherical Volume Charge - Find potential everywhere.



$\Rightarrow$ Cannot solve with Laplace since $\rho \neq 0$

Ex Thin spherical shell of charge of radius a and charge density σ.



The region inside and outside satisfy Laplace's eqn

General Solution

inside $(r<a)$ $\quad V_i = A_1 + \frac{B_1}{r}$

outside $(r>a)$ $\quad V_o = A_2 + \frac{B_2}{r}$

Boundary Conditions

I. $V_o(\infty) = 0 \implies A_2 = 0$

II. $V_i(0)$ finite $\implies B_1 = 0$

III. Continuous at $r = a \implies$

$$V_i(a) = A_1 = V_o(a) = \frac{B_2}{a}$$

$$\implies B_2 = A_1 a$$

IV. Correct charge density at boundary
$\implies$ Electrostatic boundary condition

$$\left.\frac{\partial V_o}{\partial r}\right|_a - \left.\frac{\partial V_i}{\partial r}\right|_a = \frac{-\sigma}{\varepsilon_0}$$

$$\left.\frac{\partial V_o}{\partial r}\right|_a = -\frac{B_2}{a^2}$$

$$\left.\frac{\partial V_i}{\partial r}\right|_a = 0$$

$$\implies -\frac{B_2}{a^2} + 0 = \frac{-\sigma}{\varepsilon_0}$$

$$B_2 = \frac{\sigma a^2}{\varepsilon_0}$$

$$A_1 = \frac{B_2}{a} = \frac{a\sigma}{\varepsilon_0}$$

Full Solution

$$r<a: \quad V_i = \frac{a\sigma}{\varepsilon_0}$$

$$r>a: \quad V_o = \frac{a^2\sigma}{\varepsilon_0 r} = \frac{4\pi a^2\sigma}{4\pi\varepsilon_0 r}$$

$$= \frac{Q}{4\pi\varepsilon_0 r} \quad \checkmark$$

Units $\quad [V] = \frac{Nm}{C} \quad \checkmark$

Field

$$\vec{E}_i = -\nabla V_i = 0$$

$$\vec{E}_o = -\nabla V_o = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

Azimuthal Symmetry (B.C. independent of ϕ)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

Separation of Variables

$$V(r,\theta) = R(r)\,T(\theta)$$

Solutions $\quad 1, \; 1/r, \; \ln(\tan\theta/2)$

$$r^n P_n(\cos\theta) \qquad r^{-(n+1)} P_n(\cos\theta)$$

Legendre Polynomial $P_n(x)$

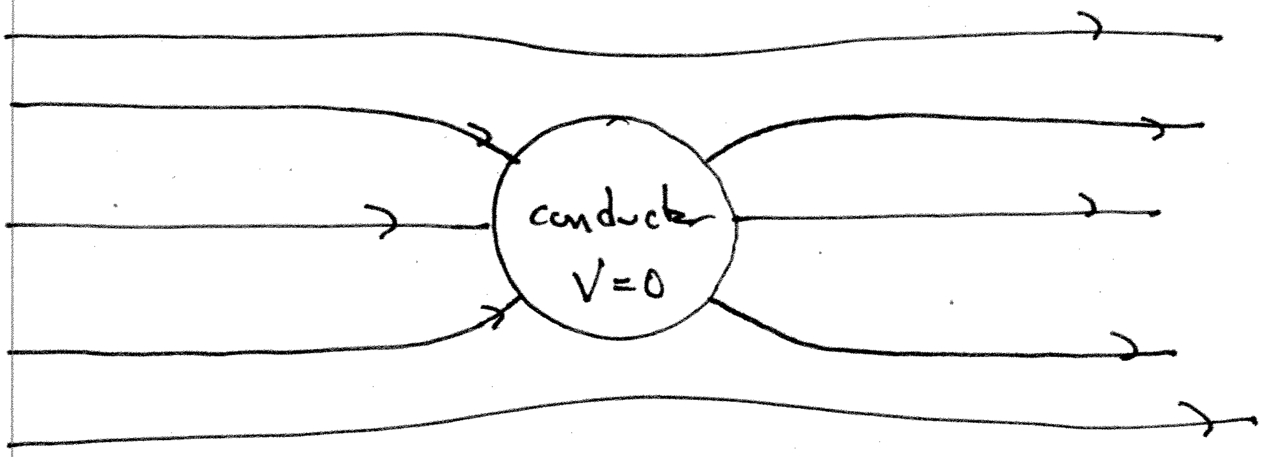
$$P_0 = 1, \quad P_1 = x, \quad P_2 = \tfrac{1}{2}(3x^2-1)$$

Orthogonality

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{nm}$$

$$\int_{-1}^{1} P_n(\cos\theta)P_m(\cos\theta)\,d(\cos\theta) = \frac{2}{2n+1}\delta_{nm}$$

Ex Conducting Sphere in Applied Uniform Field $\vec{E} = E_0 \hat{x}$



Choose the zero of potential at the conductor

Bandory Conditions

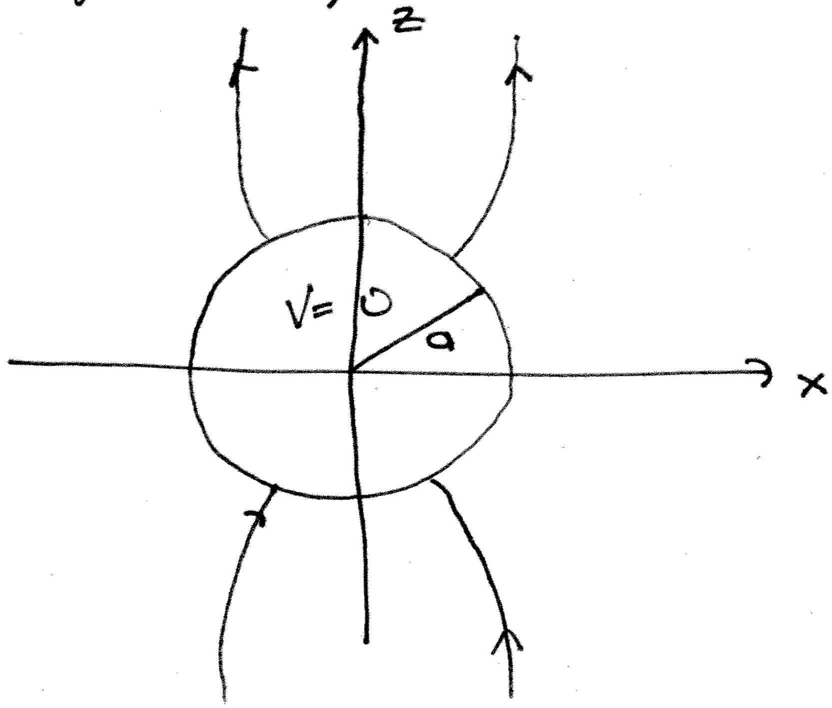
I. $\vec{E} \to E_0 \hat{x}$ as $x \to \pm\infty$

II $V_{sphere} = 0$

III $Q_{sphre} = 0$

Solution Spherical problem independent of ϕ.

Exchange axes, $x \to z$



Boundary Condition in Spherical Coordinates

$$\vec{E} = E_0 \hat{z} \implies V \to -E_0 z + C$$

From Griffiths Cover

$$z = r\cos\theta$$

$$V \to -E_0 r \cos\theta + C$$

To use orthogonality, we need θ in terms of P_l

$$P_1(x) = x \implies P_1(\cos\theta) = \cos\theta$$

$$V \to -E_0 r P_1(\cos\theta) + C$$

General Solution $r > a$

$$V(r,\theta) = \sum_{n=0}^{\infty} A_n r^{-(n+1)} P_n(\cos\theta)$$

$$+ B_n r^n P_n(\cos\theta)$$

Look for Explosions

As $r \to \infty$, $r^n \to \infty$, so to match

$V \to -E_0 r P_1(\cos\theta) + C$, r^n $n \geq 2$

must vanish so only B_0 and B_1 survive

$$B_0 = C \qquad B_1 = -E_0$$

$$V(r,\theta) = \sum_n A_n r^{-(n+1)} P_n(\cos\theta)$$

$$- E_0 r P_1(\cos\theta) + C$$

All $r^{-(n+1)}$ terms go to zero as $r \to \infty$

so they are fine.

Apply $r=a$ Boundary Condition

$$V(a,\theta) = 0 \quad \sum_{n=0}^{\infty} A_n a^{-(n+1)} P_n(\cos\theta)$$
$$- E_0 a P_1(\cos\theta) + C$$

Method I

The $P_i(x)$ are orthogonal functions so you can treat them like unit vectors (which they are in function space if we normalized them).

$\Rightarrow$ Coefficents of P_i must be independently zero.

P_0 : $\dfrac{A_0}{a} - C = 0$

P_1 : $A_1 a^{-(1+1)} - E_0 a = 0$

$$A_1 = a^3 E_0$$

P_2 : $A_2 = 0$

$P_{n>2}$: $A_n = 0$

Now a bit of physics, no work is required to move a charge in from ∞ along the x axis

$$V(\infty, \tfrac{\pi}{2}) = -E_0 r \cos\tfrac{\pi}{2} + C = V(a) = 0$$

$$\Rightarrow C = 0.$$

$$\Rightarrow A_0 = 0$$

Solution

$$V(r,\theta) = \frac{a^3 E_0}{r^2} P_1(\cos\theta) - E_0 r P_1(\cos\theta)$$

$$= \left(\frac{a^3 E_0}{r^2} - E_0 r\right) \cos\theta$$

Method II Fourier's Trick

$$V(a,\theta) = 0 = \sum_n A_n a^{-(n+1)} P_n(\cos\theta) - E_0 a P_1(\cos\theta)$$

$$\int_{-1}^{1} 0 \cdot P_m(\cos\theta)\, d\cos\theta = 0$$

$$= \sum A_n a^{-(n+1)} \int_{-1}^{1} P_m(\cos\theta) P_n(\cos\theta)\, d(\cos\theta)$$

$$+ \int_{-1}^{1} (-E_0 a) P_m(\cos\theta) P_1(\cos\theta)\, d\cos\theta$$

If $m = 0, \geq 2$

$$0 = A_m a^{-(m+1)} \cdot \frac{2}{2m+1} \Rightarrow A_m = 0$$

If $m = 1$,

$$0 = \frac{A_1}{a^2} \cdot \frac{2}{2\cdot1+1} - E_0 a \frac{2}{2\cdot1+1}$$

$$\Rightarrow A_1 = E_0 a^3$$

For the solution to be physical, $Q = 0$ and $E \perp$ surface of conductor.

Find $\vec{E}$

$$\vec{E} = -\nabla V$$

$$= -\left(\left(\frac{\partial V}{\partial r}\right) \hat{r} + \frac{1}{r}\left(\frac{\partial V}{\partial \theta}\right) \hat{\theta} \right)$$

$$= \left(E_0 \cos\theta + \frac{2E_0 a^3 \cos\theta}{r^3} \right) \hat{r}$$

$$- \left(\frac{E_0 r \sin\theta}{r} - \frac{E_0 a^3 \sin\theta}{r^3} \right) \hat{\theta}$$

At the surface of the conductor,

$$\vec{E}(a, \theta) = (E_0 \cos\theta + 2E_0 \cos\theta)\, \hat{r}$$

$$- (E_0 \sin\theta - E_0 \sin\theta)\, \hat{\theta}$$

$$= 3E_0 \cos\theta\, \hat{r}$$

which is $\perp$ to conductor's surface.

Surface Charge Density (Gaussian Pillbox)

$$\phi = \vec{E}_0 \cdot \hat{r} A - \underset{\overset{\shortparallel}{0}}{\vec{E}_i \cdot \hat{r} A} = \frac{\sigma}{\varepsilon_0} A$$

$$3 E_0 \cos\theta = \frac{\sigma}{\varepsilon_0}$$

$$\sigma = 3 \varepsilon_0 E_0 \cos\theta$$

$\Rightarrow$ More charge at the poles and the correct sign charge.

<u>Total Charge</u>

$$Q = \int \sigma \, da \qquad da = (R\,d\theta)(R\sin\theta\, d\phi)$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, (R^2 \sin\theta)(3 E_0 \varepsilon_0 \cos\theta)$$

$$= 6\pi R^2 E_0 \varepsilon_0 \int_0^{\pi} \underset{\overset{\shortparallel}{\frac{1}{2}\sin 2\theta}}{\sin\theta\cos\theta} \, d\theta = 0$$

Full Solution Laplace's Eqn

$$V(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[A_{\ell m} r^{\ell} + B_{\ell m} r^{-(\ell+1)} \right]$$

$$\times \, Y_\ell^m(\phi,\theta)$$

Spherical Harmonics

$$Y_\ell^m = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \, P_\ell^m(\cos\theta) \, e^{im\phi}$$

Associated Legendre Polynomial P_ℓ^m

Orthogonality

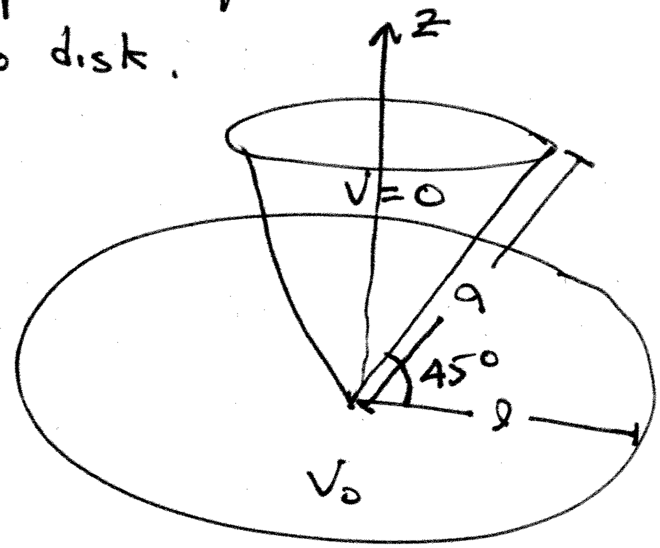
$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, Y_{\ell'}^{*m'} Y_\ell^m = \delta_{\ell\ell'} \delta_{mm'}$$

The trivial solutions to the full eqn are

$1, \; 1/r, \; \ln(\tan \theta/2), \; \phi$

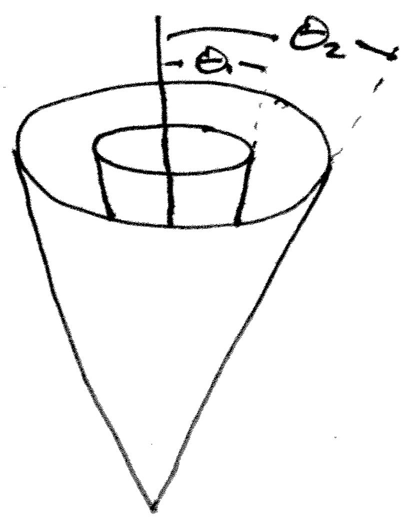
and come in handy for some odd systems

<u>Ex</u> Compute capacitance of cone at 45° angle to disk.



Assume no fringing.

This is similar to the following



Try $V(\theta) = A \ln\left(\tan\frac{\theta}{2}\right) + B$

Boundary Conditions

$V(45°) = 0 \qquad\qquad V(90°) = V_0$

$\Rightarrow$ We automatically satisfy $\vec{E} \perp$ surface because the surfaces are equipotentials

$$V(45°) = A \ln\left(\tan\frac{\pi}{8}\right) + B = 0$$

$$V(90°) = A \ln\left(\underbrace{\tan \pi/4}_{=1}\right) + B = V_0$$

$$= 0 + B = V_0$$

$$B = V_0$$

$$A = \frac{-B}{\ln(\tan \pi/8)} = \frac{-V_0}{\ln(\tan \pi/8)}$$

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$$V(\theta) = -V_0 \left(\frac{\ln(\tan \theta/2)}{\ln(\tan \pi/8)} - 1 \right)$$

Calculate Electric Field

$$\vec{E} = -\nabla V = -\frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta}$$

$$* \quad \frac{\partial}{\partial \theta} \ln \tan \theta/2$$

$$= \frac{1}{\tan \theta/2} \cdot \frac{\partial}{\partial \theta} \tan \theta/2$$

$$= \frac{1}{\tan \theta/2} \cdot \frac{1}{2} \cdot \sec^2 \theta/2$$

$$= \frac{1}{2} \frac{\cos \theta/2}{\sin \theta/2} \cdot \frac{1}{\cos^2 \theta/2}$$

$$= \frac{1}{2 \sin \theta/2 \cos \theta/2} = \frac{1}{\sin \theta}$$

$$\vec{E} = \frac{V_0}{r \sin \theta} \cdot \frac{1}{\ln(\tan \pi/8)} \hat{\theta}$$

Let $\gamma = \ln(\tan \pi/8) = -0.88\ldots$

so field points in the $-\hat{\theta}$ direction

which is correct.

$$\vec{E} = \frac{V_0}{r\gamma \sin\theta}\,\hat{\theta}$$

<u>Compute the total charge on the flat disk</u>

<u>Charge Density</u>

$$\sigma = \varepsilon_0 |\vec{E}| \qquad \theta = 90^\circ$$

$$= -\frac{V_0}{\gamma r}\varepsilon_0 \qquad \text{(Force charge positive)}$$

<u>Total Charge</u>

$$Q = \int_0^{2\pi} d\phi \int_0^{l} dr\, \sigma \qquad da = (dr)(r\,d\phi)$$

$$= -\frac{V_0 \varepsilon_0}{\gamma} \int_0^{2\pi} d\phi \int_0^{l} dr$$

$$= \frac{-V_0 2\pi l \varepsilon_0}{\gamma}$$

Capacitance

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{V_0}$$

$$= \frac{-2\pi l \varepsilon_0}{\gamma} = \frac{-2\pi l \varepsilon_0}{\ln(\tan \pi/8)}$$

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