Laplace's Eqn (Cartesian)

Many times we will solve for the potential for systems with substantial charge free areas. In these regions the potential obeys Laplace's equation.

$$\nabla^2 V = 0$$

We will work out the solution to Laplace's eqn in cartesian, cylindrical, and spherical coordinates.

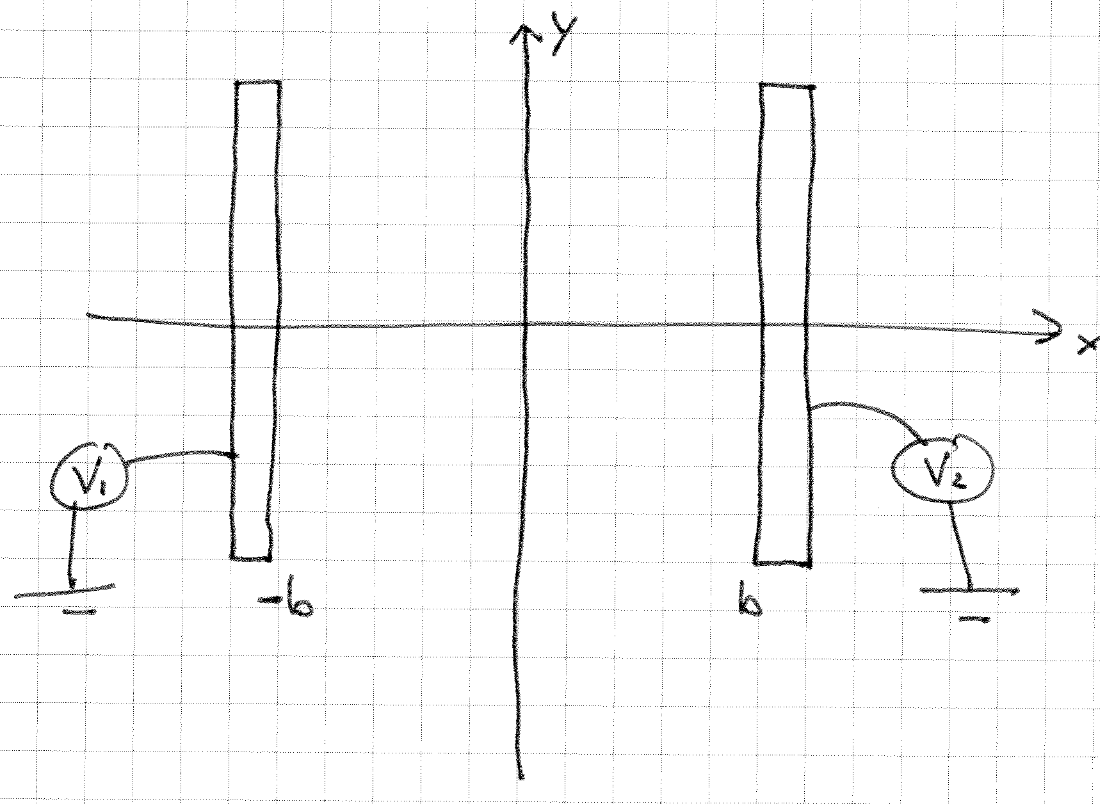
Cartesian Laplace's Eqn

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

1D – The potential is constant in the y and z directions and only changes in x direction

$$V(\vec{r}) = V(x)$$

Ex Field between two conducting plates held at V_1 and V_2.



Since $\rho = 0$ between the plates,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 0$$

Solutions $V = 1$, $V = x$

General Solution

$$V(x) = a_1 + a_2 x$$

$$a_i = \text{constant}$$

Apply Boundary Conditions

$$V(b) = V_2 = a_1 + a_2 b$$

$$V(-b) = V_1 = a_1 - a_2 b$$

$$V_1 + V_2 = 2a_1$$

$$a_1 = \frac{V_1 + V_2}{2}$$

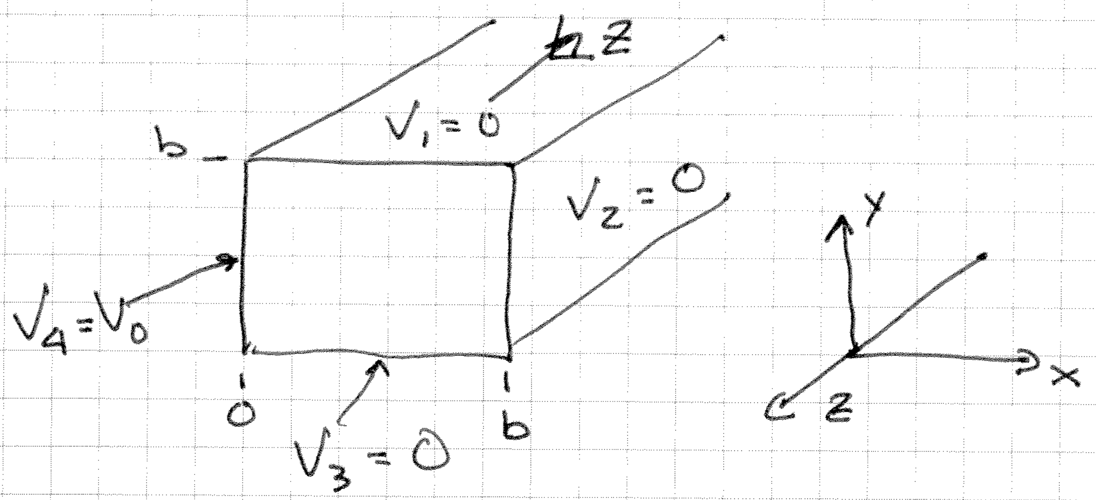
$$V_2 - V_1 = 2a_2 b$$

$$a_2 = \frac{V_2 - V_1}{2b}$$

Solution $V(x,y,z) = \dfrac{V_1 + V_2}{2} + \dfrac{V_2 - V_1}{2b} x$

2D Cartesian

Potential does not change in z direction $\Rightarrow$ infinitely long channel



Boundary Conditions (example)

$$V(b, y, z) = V_2 = 0$$

$$V(0, y, z) = V_4 = V_0$$

$$V(x, 0, z) = V_3 = 0$$

$$V(x, b, z) = V_1 = 0$$

Laplace's Eqn

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Separation of Variables

Propose a separable solution

$$V(x, y) = X(x)\, Y(y)$$

that is a product of functions of one variable.

- Does not always work
- Does not yield all solutions.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = Y(y)\frac{\partial^2 X(x)}{\partial x^2} + X(x)\frac{\partial^2 Y(y)}{\partial y^2} = 0$$

Divide by $V = XY$

$$\underbrace{\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}}_{= k^2} + \underbrace{\frac{1}{Y(y)}\frac{\partial^2 Y}{\partial y^2}}_{= -k^2} = 0$$

- Since each depends on different variables, each must separately be constant.
- The choice of $\pm k^2$ depends on boundary conditions.

With my choice

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = k^2$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = -k^2$$

$$\frac{d^2 X(x)}{dx^2} - k^2 X(x) = 0 \qquad \Bigg| \qquad \frac{d^2 Y(y)}{dy^2} + k^2 Y(y) = 0$$

Solutions

$$X(x) = \begin{matrix} e^{kx} \\ e^{-kx} \end{matrix} \qquad \Bigg| \qquad Y(y) = \begin{matrix} \sin ky \\ \cos ky \end{matrix}$$

The boundary condition may not allow all values of k.

Impose y Boundary Condition

$$Y(0) = 0 \qquad Y(b) = 0$$

$$Y(0) = 0 \Rightarrow Y(y) = \sin kx$$

$Y(b) = 0 \implies \sin kb = 0$

$$\implies kb = n\pi$$

$$\implies k = \frac{n\pi}{b} \equiv k_n$$

where n integer.

At this point any ~~$e^{\pm k_n x}$ sin~~

$$e^{\pm k_n x} \sin k_n y$$

satisfies Laplace's eqn and the y boundary condition.

The general solution is

$$V(x,y) = \sum_{n=1}^{\infty} \sin k_n y \left(C_n^+ e^{k_n x} + C_n^- e^{-k_n x} \right)$$

$C_n^\pm$ constants to be determined

Impose Boundary Conditions

$$V(0,y) = V_0 = \sum_{n=1}^{\infty} (c_n^+ + c_n^-) \sin k_n y$$

$$V(b,y) = 0 = \sum_{n=1}^{\infty} (c_n^+ e^{k_n b} + c_n^- e^{-k_n b}) \sin k_n y$$

To find $c_n^\pm$ we need methods to pick these sums apart.

General Properties of Solution to Laplace's Eqn

Completeness - The set $\{f_n\}$ is complete if any function $f(x)$ satisfying $\nabla^2 f = 0$ and the boundary conditions can be written as a linear combination of $\{f_n\}$

$$f(x) = \sum_{n=1}^{\infty} a_n f_n(x)$$

where a_i are constants

We already implicitly used completeness to write the general solution as a sum.

Orthogonality

$$\int_0^b \sin k_n x \sin k_m x \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{b}{2} & \text{if } m = n \end{cases}$$

$$= \frac{b}{2} \delta_{mn}$$

Kronecker Delta

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

$\Rightarrow$ The functions $\{\sin k_n x\}$ behave as orthogonal vectors.

$\Rightarrow$ $\{\sin k_n x\}$ are linearly independent

$\Rightarrow$ We cannot make one $\sin k_n x$ out of combinations of the other.

Fourier's Trick To extract a_m from the series

$$f(x) = \sum_n a_n \sin k_n x = \sum_n a_n f_n(x)$$

Multiple by $f_m(x)$ and integrate,

$$f_m(x) = \sin k_m x.$$

$$\int_0^b f_m(x) f(x)\, dx = \int_0^b f(x) \sin k_m x\, dx$$

$$= \sum_n \int_0^b a_n \sin k_n x \sin k_m x\, dx$$

$$= \sum_n a_n \frac{b}{2} \delta_{nm} = \frac{b}{2} a_m$$

$$a_m = \frac{2}{b} \int_0^b f(x) \sin k_m x\, dx$$

For our channel,

$$V(b,y) = 0 = \sum_{n=1}^{\infty} \left(C_n^+ e^{k_n b} + C_n^- e^{-k_n b} \right) \sin k_n y$$

Since $\sin k_n y$ are linearly independent, the coefficients of each must be zero.

$$C_n^+ e^{k_n b} + C_n^- e^{-k_n b} = 0$$

$$C_n^- = -C_n^+ e^{2 k_n b}$$

$$V(0,y) = \sum_{n=1}^{\infty} C_n^+ \left(1 - e^{2 k_n b} \right) \sin k_n y = V_0$$

<u>Fourier's Trick</u> Multiply by $\sin k_m y$ and integrate

$$\int_0^b V_0 \sin k_m y \, dy = \sum_{n=1}^{\infty} C_n^+ \left(1 - e^{2 k_n b} \right) \times \underbrace{\int_0^b \sin k_m y \sin k_n y \, dy}_{\frac{b}{2} \delta_{nm}}$$

$$\int_0^b V_0 \sin k_m y \, dy = \sum_{n=1}^{\infty} c_n^+ \left(1 - e^{2k_n b}\right) \frac{b}{2} \delta_{nm}$$

$$= c_m^+ \frac{b}{2} \left(1 - e^{2k_m b}\right)$$

$$\int_0^b V_0 \sin k_m y \, dy = V_0 \int_0^b \sin \frac{m\pi y}{b} \, dy$$

$$= V_0 \frac{b}{m\pi} \int_0^{m\pi} \sin u \, du$$

$$= \frac{V_0 b}{m\pi} \left(-\cos u\right) \Big|_0^{m\pi}$$

$$= \begin{cases} 0 & m \text{ even} \\ \dfrac{2V_0 b}{m\pi} & m \text{ odd} \end{cases}$$

$$= c_m^+ \frac{b}{2} \left(1 - e^{2k_m b}\right)$$

$$c_m^+ = \begin{cases} 0 & m \text{ even} \\ \dfrac{4V_0}{m\pi \left(1 - e^{2k_m b}\right)} & m \text{ odd} \end{cases}$$

Full Solution

$$V(x,y) = \sum_n \left(c_n^+ e^{k_n x} + c_n^- e^{-k_n x}\right) \sin k_n y$$

$$= \sum_{n\,odd} \frac{4 V_0 \sin k_n y}{n\pi\left(1 - e^{-2k_n b}\right)} \left(e^{k_n x} - e^{2k_n b} e^{-k_n x}\right)$$

* From this $\vec{E} = -\nabla V$

* The charge density on one of the conducting faces can be found as

$$\sigma = \varepsilon_0 E_n = -\varepsilon_0 \frac{\partial V}{\partial n}$$

Ex Charge on plate in x-z plane

$$\sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial x}\right|_{x=0}$$

$$= \sum_{n\,odd} \frac{-4\varepsilon_0 V_0 \sin k_n y}{n\pi\left(1 - e^{2k_n b}\right)} \left(k_n + k_n e^{2k_n b}\right)$$

For each coordinate system, there will be some special functions that are useful but are not part of the complete set.

For 2D Cartesian, the special (trivial) solutions are $1, x, y$

Laplace's Eqn - Cartesian - 3D

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Separation of Variables

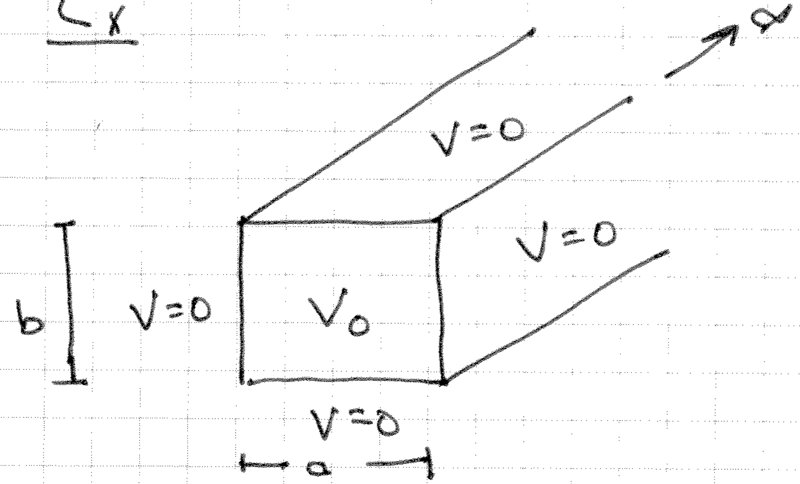
$$V = X(x)\,Y(y)\,Z(z)$$

$$0 = \underbrace{\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x}}_{= -k^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y(y)}{\partial y^2}}_{= -l^2} + \underbrace{\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}}_{= w^2}$$

$$-k^2 - l^2 + w^2 = 0$$

$\Rightarrow$ Signs chosen to meet boundary conditions

Ex



Semi-infinite channel with one end held at V_0 and sides grounded.

Solutions (after boundary conditions)

$$X(x) = \sin k_n x \qquad k_n = \frac{n\pi}{a}$$

$$Y(y) = \sin l_m y \qquad l_m = \frac{m\pi}{b}$$

$$Z(z) = e^{wz} , e^{-wz}$$

$\Rightarrow$ Discard e^{wz} because blows up at ∞

$$V(x,y,z) = \sum_{mn} A_{nm} \sin k_n x \sin l_m y \; e^{-(k_n^2 + l_m^2)^{1/2} z}$$