Maxwell's Equations for Magnetism

Let's calculate $\nabla \cdot \vec{B}$ and $\nabla \times \vec{B}$ from the Biot-Savart Law.

$$\nabla \cdot \vec{B} = \frac{\mu_0}{4\pi} \int_V \nabla \cdot \left(\frac{\vec{J}(\vec{r}\,') \times \hat{r}''}{r''^2} \right) d\tau'$$

Identity

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\nabla \cdot \left(J(\vec{r}\,') \times \left(\frac{\hat{r}''}{r''^2} \right) \right) = \frac{\hat{r}''}{(r'')^2} \cdot \nabla \cdot \vec{J}(\vec{r}\,')$$

$$- \vec{J}(\vec{r}\,') \cdot \nabla \times \left(\frac{\hat{r}''}{r''^2} \right)$$

$\nabla \cdot \vec{J}(\vec{r}\,') = 0$ because $\vec{J}$ depends on primed variables, but ∇ takes derivatives with respect to unprimed variables.

$\nabla \times \left(\frac{\hat{r}''}{r''^2} \right) = 0$ because the curl of a point charge field is zero.

No Magnetic Monopoles

$$\nabla \cdot \vec{B} = 0 \quad \longleftrightarrow \quad \oint_S \vec{B} \cdot d\vec{a} = \Phi_m = 0$$

$\Rightarrow$ Net magnetic flux out of closed surface is zero.

$\Rightarrow$ No isolated magnetic charge

$\Rightarrow$ Magnetic field lines are closed curves.

Now look at $\nabla \times \vec{B}$

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int_V \nabla \times \left(\vec{J}(\vec{r}\,') \times \frac{\hat{r}''}{(r'')^2} \right) d\tau'$$

Identity

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

$$+ \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A})$$

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int_V \Bigg[\underbrace{\left(\frac{\hat{r}''}{(r'')^2} \cdot \nabla \right) \vec{J}(\vec{r}')}_{1} - \underbrace{(J \cdot \nabla) \left(\frac{\hat{r}''}{r''^2} \right)}_{2}$$

$$+ \underbrace{\vec{J} \left(\nabla \cdot \left(\frac{\hat{r}''}{r''^2} \right) \right)}_{3} - \underbrace{\frac{\hat{r}''}{(r'')^2} \left(\nabla \cdot \vec{J}(\vec{r}') \right)}_{4} \Bigg] d\tau'$$

1 + 4 are zero because unprimed derivative of primed coordinates.

As before, $\nabla \cdot \left(\frac{\hat{r}''}{r''^2} \right) = 4\pi \delta^3(\vec{r}'')$

$$3 = \vec{J}(\vec{r}') \, 4\pi \delta^3(\vec{r}'')$$

<u>Now work on 2</u>

In $\vec{r}''$, $\hat{r}''$, r'' the variables x, x', y, y', z, z' only appear in the combinations $x-x'$, $y-y'$, $z-z'$.

So the effect of exchanging $x \leftrightarrow x'$ etc is to change signs of the gradient

$$(\vec{J} \cdot \nabla) \left(\frac{\hat{r}''}{r''^2} \right) = -(\vec{J} \cdot \nabla') \left(\frac{\hat{r}''}{r''^2} \right)$$

$$= -(\vec{J} \cdot \nabla') \left(\frac{\vec{r}''}{r''^3} \right)$$

Look at components

x – component

$$\left(J(\vec{r}\,') \cdot \nabla'\right)\left(\frac{x-x'}{(r'')^3}\right)$$

Another identity

$$\nabla \cdot \left(f\vec{A}\right) = f\left(\nabla \cdot \vec{A}\right) + \left(\vec{A} \cdot \nabla\right) f$$

$$\left(\vec{A} \cdot \nabla\right) f = \nabla \cdot \left(f\vec{A}\right) - f\left(\nabla \cdot \vec{A}\right)$$

So

$$\left(J(\vec{r}\,') \cdot \nabla'\right)\left(\frac{x-x'}{(r'')^3}\right) = \nabla' \cdot \left(\frac{x-x'}{(r'')^3}\, J(\vec{r}\,')\right)$$

$$- \frac{x-x'}{(r'')^3}\left(\nabla' \cdot \vec{J}(\vec{r}\,')\right)$$

The second term is zero because $\nabla' \cdot \vec{J} = 0$ in magnetostatics.

Integrate the remaining term

$$\int_V \nabla' \cdot \left(\frac{x - x'}{(r'')^3} J(\vec{r}') \right) d\tau'$$

$$= \oint_S \frac{x - x'}{(r'')^3} \vec{J}(\vec{r}') \cdot d\vec{a}$$

$$= 0$$

if we ensure V is large enough so no current passes through the surface.

So only term 3 is left.

$$\nabla \cdot \vec{B} = \frac{\mu_0}{4\pi} \int_V \vec{J}(\vec{r}')\, 4\pi\, \delta^3(\vec{r}'')\, d\tau'$$

$$= \mu_0 \vec{J}$$

Ampere's Law (Magnetostatics)

$$\nabla \times \vec{B} = \mu_0 J$$

Convert to integral form

$$\int_S \nabla \times \vec{B} \cdot d\vec{a} = \int_S \mu_0 \vec{J} \cdot d\vec{a}$$

$$\int \vec{J} \cdot d\vec{a} \equiv I_{enc}$$

Stokes

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$ Ampere's Law Integral Form

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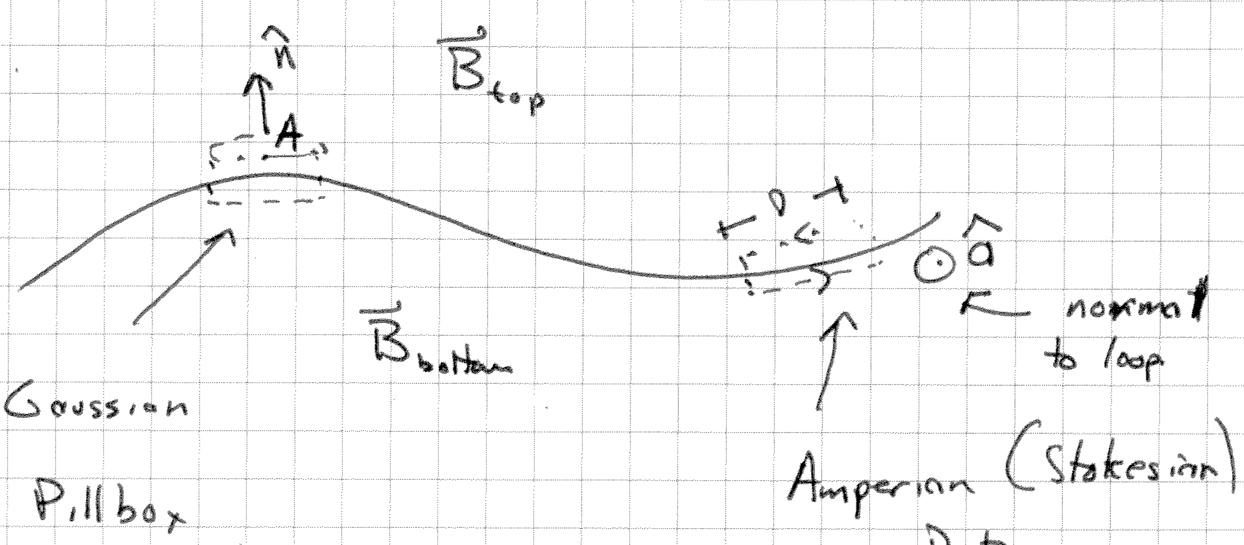
Magnetostatic Boundary Conditions



$$\Phi_m = \int \vec{B} \cdot d\vec{a} = 0 \qquad \text{(No magnetic monopoles)}$$

$$\Phi_m = \vec{B}_{top} \cdot \hat{n} A - \vec{B}_{bottom} \cdot \hat{n} A = 0$$

$$B_{top}^{\perp} = B_{bottom}^{\perp}$$

Ampere's Law

$$\oint_c \vec{B} \cdot d\vec{l} = \vec{B}_{top} \cdot \hat{t}\, l - \vec{B}_{bottom} \cdot \hat{t}\, l$$

$$= \mu_0 I_{enc} = \mu_0 \vec{K} \cdot \hat{a}$$

$\vec{K} \equiv$ surface current

$\hat{a} \equiv$ surface normal Amperian loop

In vector form we can combine these as

$$\vec{B}_{top} - \vec{B}_{bottom} = \mu_0 (\vec{K} \times \hat{n})$$