# Multipole Expansion

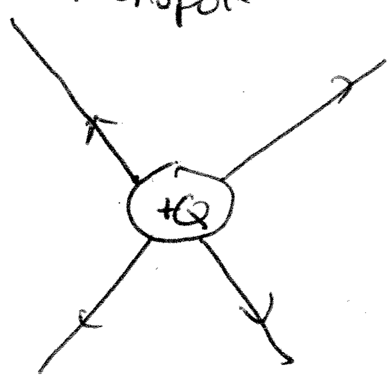
We would like to develop and expansion that allows the investigation of the fields of complicated extended objects

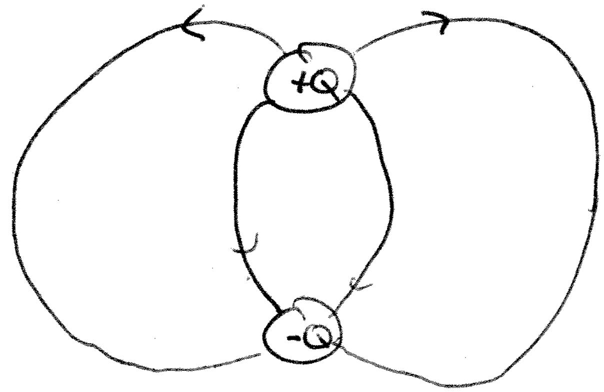
⇒ We know to leading order the field will be

$$\vec{E} = \frac{kQ_{Total}}{r^2}\hat{r} \qquad (\text{monopole term})$$

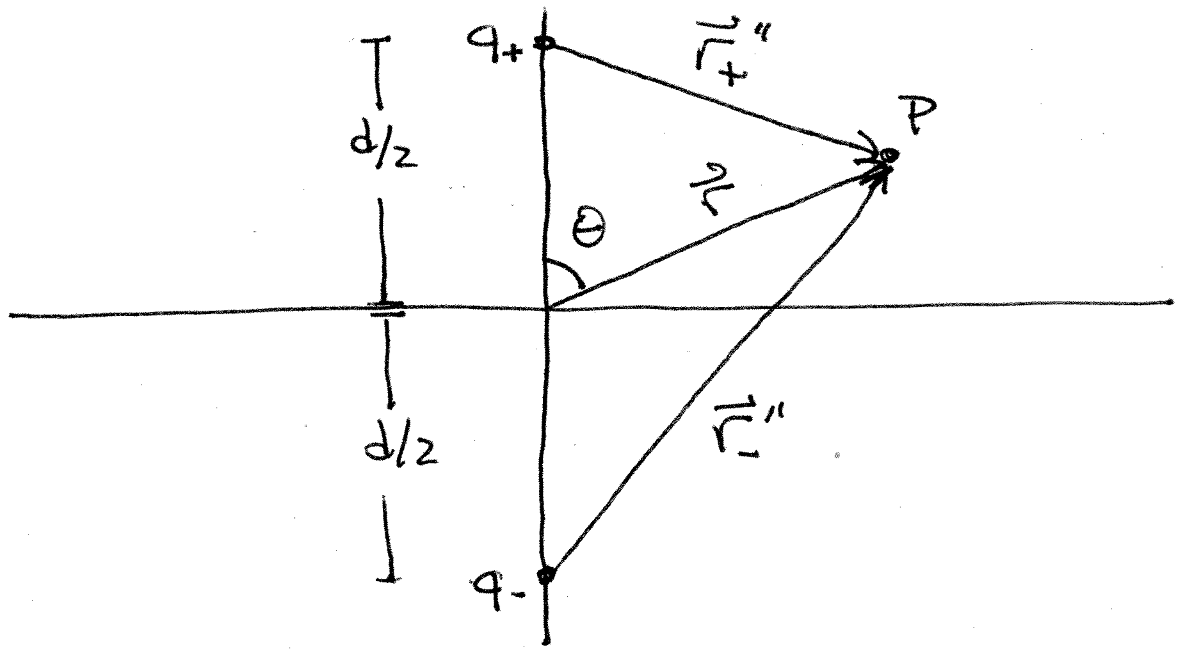
⇒ We could improve the approximation by adding a dipole term.

Monopole



# Potential of Simple Dipole




$$V(\vec{r}) = kq\left(\frac{1}{r_+''} - \frac{1}{r_-''}\right)$$

## Law of Cosines

$$r_+''^{\,2} = r^2 + \left(\frac{d}{2}\right)^2 - rd\cos\theta$$

$$r_-^{2} = r^2 + \left(\frac{d}{2}\right)^2 + rd\cos\theta$$

$$r_+'' = r^2\left(1 + \left(\frac{d}{2r}\right)^2 - \frac{d}{r}\cos\theta\right)$$

$$r_-'' = r^2\left(1 + \left(\frac{d}{2r}\right)^2 + \frac{d}{r}\cos\theta\right)$$

$$\frac{1}{r_+''} = \frac{1}{r\sqrt{1 + \left(\frac{d}{2r}\right)^2 - \frac{d}{r}\cos\theta}}$$

$$\approx \frac{1}{r\sqrt{1 - \left(\frac{d}{r}\cos\theta\right)}} \quad \text{if } \frac{d}{r} \text{ small}$$

$$\approx \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right)$$

binomial expansion

$(1+x)^n \sim 1 + nx + \cdots$

$n = -\frac{1}{2}$

$$\frac{1}{r_-''} \approx \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right)$$

$$\frac{1}{r_+''} - \frac{1}{r_-''} \approx \frac{d}{r^2}\cos\theta$$

So

$$V(\vec{r}) = \frac{Kqd\cos\theta}{r^2}$$

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Dfn Dipole Moment Vector ($\vec{p}$) - Vector points from center of $-$ to center of $+$ charge.

$$|\vec{p}| = qd$$

so $\vec{p}\cdot\hat{r} = qd\cos\theta$

$$V(\vec{r}) = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}$$

which from homework 1 gives the field

$$\vec{E} = -\nabla V = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

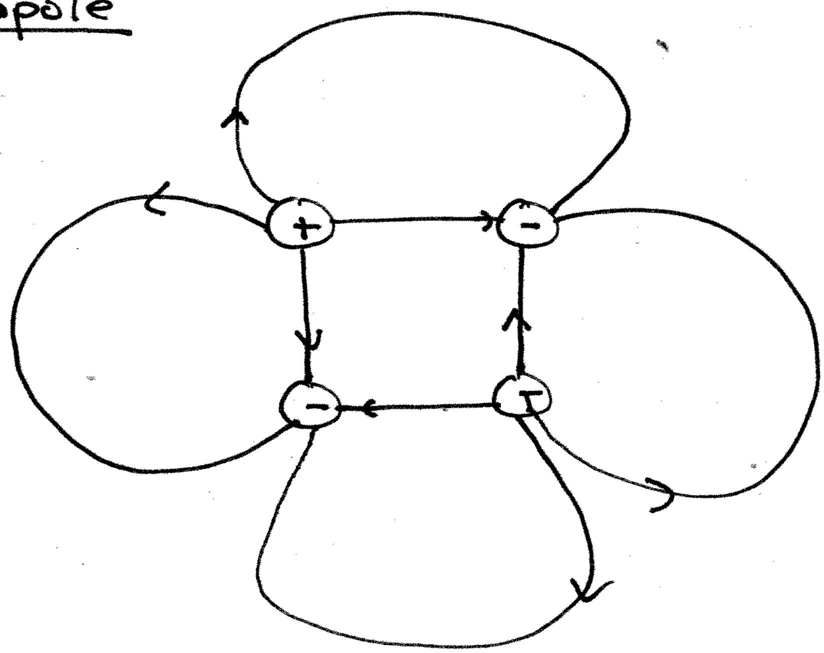
if dipole $\vec{p} = p\hat{z}$

or in general

$$\vec{E} = \frac{1}{4\pi\epsilon_0 r^3}\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right)$$

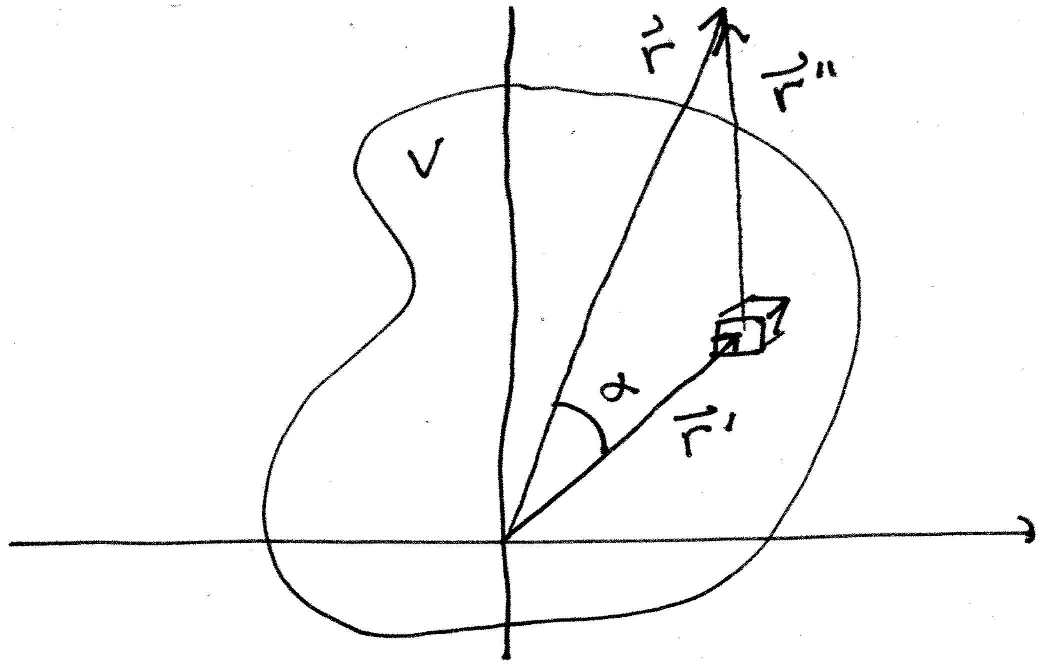
We can further improve our monopole + dipole approximation by adding higher order terms.

Quadrapole



Let's build up the general expansion

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}')\, d\tau'}{r''}$$




## Low of Cosines

$$r''^2 = r^2 + r'^2 - 2rr'\cos\alpha$$

Use the same expansion we used earlier as we let $r$ become large

$$\frac{1}{r''} = \frac{1}{r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\alpha}}$$

## Binomial Expansion

$$(1+x)^n = 1 + nx + \dots$$

For $n = -1/2$

$$\frac{1}{\sqrt{1+s}} \sim 1 - \frac{1}{2}s + \frac{3}{8}s^2 - \frac{5}{16}s^3 \ldots$$

In our case,

$$s = \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha$$

$$= \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\alpha\right)$$

$$\frac{1}{r''} = \frac{1}{r}\left(1 - \frac{r'}{2r}\left(\frac{r'}{r} - 2\cos\alpha\right)\right.$$

$$+ \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2$$

$$\left. + \ldots \right)$$

Collect Like Terms

$$\frac{1}{r''} = \frac{1}{r}\left(1 + \frac{r'}{r}\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right)\right.$$

$$\left. + \ldots \right)$$

## Recognize Legendre Polynomials

$$\frac{1}{r''} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

So

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int \rho(\vec{r}\,') \, r'^n P_n(\cos\alpha') \, d\tau'$$

Note, $\alpha'$ not constant.

## Look at it term by term

$n = 0$ : $\dfrac{1}{4\pi\varepsilon_0 r} \displaystyle\int \rho(\vec{r}\,') \, d\tau'$

<u>Monopole Moment</u> $Q_T$ = Total Charge = $\displaystyle\int_V \rho(\vec{r}\,') \, d\tau'$

<u>$n = 1$</u> $\dfrac{1}{4\pi\varepsilon_0 r^2} \displaystyle\int r'\cos\alpha' \, \rho(\vec{r}\,') \, d\tau'$

$\alpha$ - angle between $\vec{r}, \vec{r}\,'$ so

$$r'\cos\alpha = \vec{r}\,' \cdot \hat{r}$$

$$V_{dip} = \frac{\hat{r}}{4\pi\varepsilon_0 r^2} \cdot \int \vec{r}\,'\rho(\vec{r}\,')\,d\tau'$$

Dfn Dipole Moment Vector

$$\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')\,d\tau'$$

$\Longrightarrow$ Note, dipole moment of point charge not at origin is non-zero.

$\Longrightarrow$ In general, for $n \geq 0$ moments depend on origin.

$\Longrightarrow$ If $Q_T = 0$, dipole moment is independent of origin.

Ex Multipole expansion of point charge $Q$ at $\vec{r}_A = (a, 0, 0)$

$$\rho(\vec{r}) = Q\,\delta^3(\vec{r} - \vec{r}_A)$$

Monopole

$$Q_m = \int \rho(\vec{r}\,')\,d\tau' = Q$$

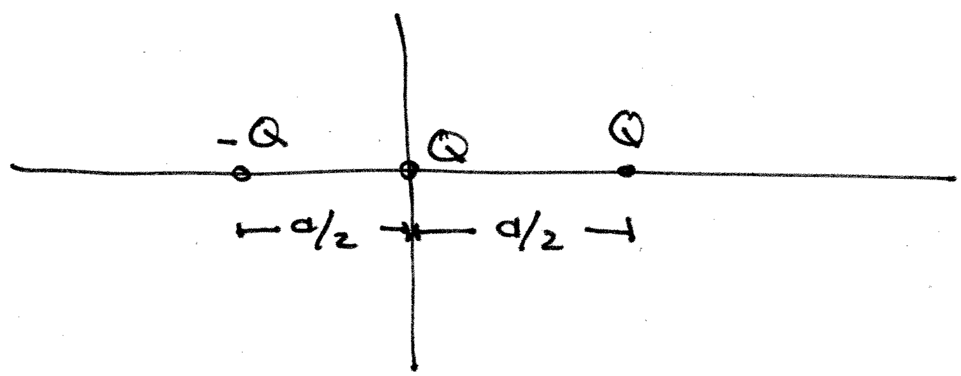
## Dipole

$$\vec{p} = \int \vec{r}\,' \rho(\vec{r}\,')\, d\tau'$$

$$= \int \vec{r}\,' Q\, \delta^3(\vec{r}\,' - \vec{r}_A)\, d\tau'$$

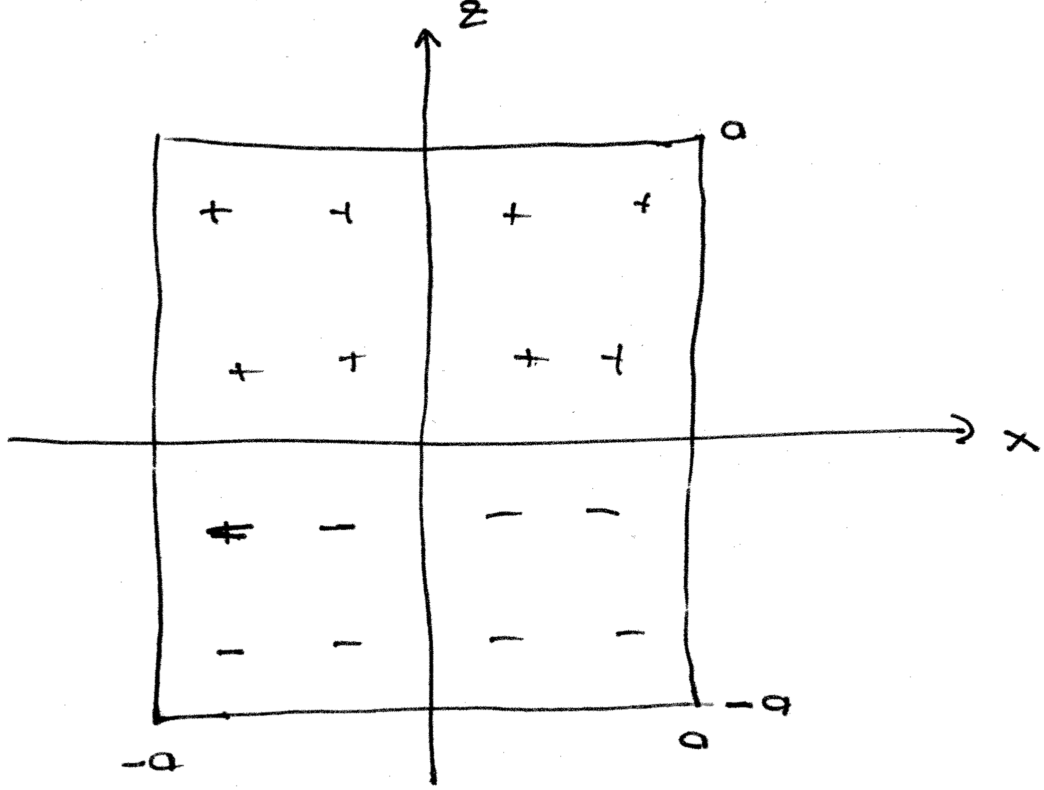
$$= Q\, \vec{r}_A$$

Addition terms work on moving Q away from origin




$\Longrightarrow$ All multipole fields centred at origin.

<u>Ex</u> Dipole moment of square region in x-z plane.




$$\sigma(z) = \begin{cases} +\sigma_0 & z > 0 \\ -\sigma_0 & z < 0 \end{cases}$$

<u>Dipole Moment</u>

$$\vec{p} = \int \vec{r}\,'\sigma(\vec{r}\,')\,d\tau'$$

Compute 3 components

$$P_x = \int_{-a}^{a}\int_{-a}^{a} x'\,\sigma(\vec{r}\,')\,dx'\,dz'$$

$$= \int_{-a}^{0} dz'\,(-\sigma_0)\underbrace{\int_{-a}^{a} dx'\,x'}_{=0} + \int_{0}^{a} dz'\,\sigma_0 \underbrace{\int_{-a}^{a} x'\,dx'}_{=0}$$

$\Rightarrow \quad P_x = 0$

$\Rightarrow \quad P_y = 0$ because $y' = 0$ on plane

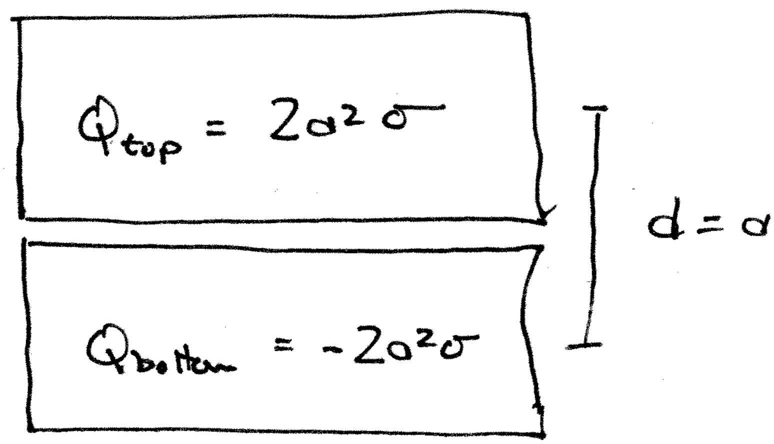
$$P_z = \int_{-a}^{a} dz' \int_{-a}^{a} dx'\; z'\,\sigma(\vec{r}\,')$$

$$= \underbrace{\int_{-a}^{0} z'\,dz'\,(-\sigma_0)}_{\frac{\sigma_0 a^2}{2}}\underbrace{\int_{-a}^{a} dx'}_{2a} + \underbrace{\int_{0}^{a} z'\,dz'\,\sigma_0}_{\frac{\sigma_0 a^2}{2}}\underbrace{\int_{-a}^{a} dx'}_{2a}$$

$$= 2a^3\sigma_0$$

$$\vec{P} = \cancel{(2a^3\sigma_0,\ 0,\ 0)} = (0, 0, 2a^3\sigma_0)$$

Does this make sense?




$$|\vec{p}| = d\,Q_{top} = (2a^2\sigma)\,a \quad \checkmark$$

Compute field at $r \gg a$.

$$\vec{E}_{dip} = \frac{p}{4\pi\varepsilon_0 r^2}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$\theta$ - angle from $z$ axis.